CHAPTER 5
TRANSVERSE WAVES IN
RODS AND BEAMS

5.1 INTRODUCTION

This chapter will be concerned with small transverse (sometimes called flexural) wave motion or vibration of linear elastic rods and beams. As in the previous two chapters we will be confining our investigation within the strength of materials viewpoint. One can therefore use the term elementary to describe our theory; however, as we will see, certain modifications will improve the initial theory and make it compare favorably with the exact theory of elasticity solution.

The passage of a transverse bending or flexural wave carries a rotation of cross-sectional planes, bending moments and bending stresses. The geometry of the deformation is much more complex than the two previous cases of longitudinal and torsional waves. Consequently it is not surprising that the equation of motion for this case is not the classical wave equation, but is a more complicated hyperbolic partial differential equation. This fact will produce dispersive behavior even though the material is linear elastic.

Additional sources for general information on this topic may be found in Nowacki [5.1], Morse [5.2], Kolsky [5.3], Clark [5.4], Rayleigh [5.5], Reisman and Pawlik [5.6], Volterra and Zachmanoglou [5.7] or Fung [5.8].
5.2 **Euler-Bernoulli Beam Theory**

We start the development by reviewing the basic assumptions from undergraduate strength of materials concerning the bending of beams. These assumptions read:

i.) the beam is prismatic and has a straight centroidal axis (which we will label the x-axis),

ii.) the beam's cross-section has an axis of symmetry (which we will label the y-axis),

iii.) all transverse loadings act in the plane of symmetry (x-y plane),

iv.) plane sections perpendicular to the centroidal axis remain plane after deformation,

v.) the material is elastic, isotropic and homogeneous,

vi.) transverse deflections are small.

The physical situation is drawn schematically in Figure 5.1. We denote the internal bending moment by $M$, the internal shear force by $V$.

![Figure 5.1](image-url)
and the external distributed loading by \( w \).

Consider the dynamic equilibrium of a beam element of length \( dx \). In using Newton's law, we must make some additional assumptions concerning the acceleration of this typical beam element. We pay special attention to these assumptions since they will determine the range of applicability of this particular theory. Consequently in addition to the previous list, the theory further assumes that:

vii.) the motion is purely translational in the \( y \)-direction, i.e., rotational effects are to be neglected,

viii.) beam elements remain rectangular during the motion.

With these assumptions, setting the vertical forces on the element equal to the mass times acceleration gives

\[
\frac{\partial y}{\partial x} = -w + \rho A \frac{\partial^2 y}{\partial t^2},
\] (5.1)

while summing moments produces

\[
\frac{\partial M}{\partial x} = V,
\] (5.2)

where \( \rho \) is the mass density of the material, \( A \) is cross-sectional area, and \( y = y(x,t) \) is the transverse motion measure. Combining (5.1) and (5.2) then yields

\[
\frac{\partial^2 M}{\partial x^2} = -w + \rho A \frac{\partial^2 y}{\partial t^2}.
\] (5.3)

From the geometry of the deformation, assumptions iv.) and vi.), and using Hooke's law \( \sigma_x = E \varepsilon_x \), assumption v.), one can show that

\[
\frac{\partial^2 y}{\partial x^2} = -\frac{M}{EI},
\] (5.4)

where \( E \) is the modulus of elasticity and \( I \) is the area moment of inertia of the cross-section about the neutral \( z \)-axis.
Finally putting (5.3) and (5.4) together gives the desired result
\[
\frac{EI}{4} \frac{\partial^4 y}{\partial x^4} + \rho A \frac{\partial^2 y}{\partial t^2} = w ,
\]
(5.5)

which is the Euler-Bernoulli beam theory equation. For the case of no external loading, \( w = 0 \), and so (5.5) becomes
\[
\frac{b^2}{4} \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0 ,
\]
(5.6)

where \( b^2 = \frac{EI}{\rho A} \). Notice that our resulting equation is not the classical wave equation, and so we can expect different solution types than seen previously.

To properly formulate a boundary value problem we also need boundary conditions for this problem type. These conditions at the ends of a beam, follow from undergraduate strength of materials. At a simply supported end, i.e., a pinned end or roller end, the deflection and moment are zero; hence
\[
y = 0 \\
\frac{\partial^2 y}{\partial x^2} = 0 .
\]
(5.7)

For a built-in or fixed end the deflection and slope are zero,
\[
y = 0 \\
\frac{\partial y}{\partial x} = 0 .
\]
(5.8)

The case of a free end dictates that the moment and shear force are zero, i.e.,
\[
\frac{\partial^2 y}{\partial x^2} = 0 \\
\frac{\partial^3 y}{\partial x^3} = 0 .
\]
(5.9)
5.3 **Separation of Variables Solution**

We now apply the separation of variables technique to equation (5.6) in order to obtain the normal modes of vibration. Substituting the form

\[ y(x,t) = X(x)T(t) \]

into (5.6) yields the two separated equations

\[
X'''' - \frac{\gamma^2}{b^2} X = 0 \\
T + \gamma^2 T = 0
\]

where \( \gamma \) is the separation constant.

The solutions to (5.10) are

\[
X(x) = C_1 \sin \frac{\gamma}{b} x + C_2 \cos \frac{\gamma}{b} x \\
\quad + C_3 \sinh \frac{\gamma}{b} x + C_4 \cosh \frac{\gamma}{b} x
\]

\[ T(t) = C_5 \sin \gamma t + C_6 \cos \gamma t \]

where \( C_1 - C_6 \) are arbitrary constants. Equation (5.11) then gives the general mode shapes, while we see from (5.12) that \( \gamma \) corresponds to the frequency of vibration.

To be more specific, consider as an example the simply supported beam shown in Figure 5.2. The boundary conditions for this beam

![Figure 5.2](image-url)
are
\[ y(0,t) = y(L,t) = 0 \]  \hspace{1cm} (5.13)
\[ \frac{\partial^2 y(0,t)}{\partial x^2} = \frac{\partial^2 y(L,t)}{\partial x^2} = 0 \]

The boundary conditions at x = 0 with form (5.11) imply that
\[ C_2 + C_4 = 0 \]
\[ -C_2 + C_4 = 0 \]

and hence
\[ C_2 = C_4 = 0 \]  \hspace{1cm} (5.14)

Applying the conditions at x = L gives
\[ C_1 \sin \sqrt{\frac{Y}{b}} L + C_3 \sinh \sqrt{\frac{Y}{b}} L = 0 \]
\[ -C_1 \sin \sqrt{\frac{Y}{b}} L + C_3 \sinh \sqrt{\frac{Y}{b}} L = 0 \]  \hspace{1cm} (5.15)

The only nontrivial solution of (5.15) exists when the determinant of the coefficients vanishes, which implies that
\[ \sin \sqrt{\frac{Y}{b}} L \sinh \sqrt{\frac{Y}{b}} L = 0 \]  \hspace{1cm} (5.16)

This is the frequency equation for this case.

Relation (5.16) then says that either \( \sin \sqrt{\frac{Y}{b}} \ell = 0 \) or \( \sinh \sqrt{\frac{Y}{b}} \ell = 0 \).

Clearly the latter case must be ruled out because it would imply that \( \gamma = 0 \) and hence would give the trivial solution. Therefore \( \sin \sqrt{\frac{Y}{b}} \ell = 0 \), and so \( \sqrt{\frac{Y}{b}} \ell = n\pi \) for \( n = 1, 2, 3, \ldots \). Consequently our eigenvalues or frequencies are
\[ \gamma_n = \frac{n^2 \pi^2}{L^2} b = \frac{n^2 \pi^2}{L^2} \sqrt{\frac{EI}{\rho A}} \]  \hspace{1cm} (5.17)
It is interesting to generalize a bit on this result. The fact of the matter is that for free vibrations of finite-length beams only certain frequencies are allowable.

With \( \sin \sqrt{\frac{\gamma}{b}} \lambda = 0 \), then (5.15) implies that \( C_3 = 0 \), and so our mode form is simply

\[
X_n = C_1 \sin \frac{n\pi x}{\lambda}.
\]  

(5.18)

So using superposition our final solution form is

\[
y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\lambda} \left( A_n \sin \gamma_n t + B_n \cos \gamma_n t \right),
\]  

(5.19)

with the constants \( A_n \) and \( B_n \) to be determined from the initial conditions. The mode forms for this case are shown in Figure 5.3 for \( n = 1, 2, 3, 4, 5 \).

![Figure 5.3](image)

Other problems with different boundary conditions are handled in the same manner and are left as exercises.
5.4 **Integral Transform Solution: Infinite Beam**

As a further example consider the free vibrations of a beam of infinite extent along the x-axis. Assume this beam is released from rest with zero velocity from a prescribed position, \( f(x) \). The boundary value problem is then equation (5.6) with initial conditions

\[
y(x, 0) = f(x) \\
y(x, 0) = 0 , \quad -\infty < x < \infty .
\] (5.20)

We solve this problem by using the method of integral transforms whose properties were outlined in Chapter 1. First taking the Laplace transform with respect to the time \( t \) of equation (5.6), gives

\[
b^2 \frac{d^4}{dx^4} y + s^2 \frac{d^2}{dx^2} y = sf(x) ,
\] (5.21)

where \( \ddot{y} = \ddot{y}(x, s) \). Next take the Fourier transform with respect to \( x \) of (5.21) to get

\[
(b^2 \alpha^4 + s^2) \tilde{y} = \tilde{sf}(\alpha) ,
\] (5.22)

where \( \tilde{y} = \tilde{y}(\alpha, s) \). Relation (5.22) is now an algebraic equation for \( \tilde{y} \) which is easily solved to get

\[
\tilde{y}(\alpha, s) = \frac{\tilde{sf}(\alpha)}{b^2 \alpha^4 + s^2} .
\] (5.23)

Being free to take the inverse transforms in either order, we take the inverse Laplace transform first. From tables

\[
L^{-1}\left\{ \frac{s}{b^2 \alpha^4 + s^2} \right\} = \cos \alpha^2 bt ;
\]

therefore,

\[
\tilde{y}(\alpha, t) = \tilde{f}(\alpha) \cos \alpha^2 bt .
\] (5.24)
Taking the inverse Fourier transform using the inversion integral (1.81), gives

\[ y(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\alpha) \cos^2 bt \ e^{i\alpha x} \ d\alpha . \]  

(5.25)

But this result can be improved upon by using the convolution property (1.80) on relation (5.24) to get

\[ y(x,t) = \int_{-\infty}^{\infty} f(x-\xi) \mathcal{F}^{-1}\{\cos^2 bt\}_{\xi} \ d\xi . \]  

(5.26)

Now

\[ \mathcal{F}^{-1}\{\cos^2 bt\}_{\xi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos^2 bt \ e^{i\alpha \xi} \ d\alpha \]

\[ = \frac{1}{2\sqrt{2\pi bt}} \left[ \cos \frac{\xi^2}{4} bt + \sin \frac{\xi^2}{4} bt \right] \]

(5.27)

\[ = \frac{1}{2\sqrt{\pi bt}} \cos \left( \frac{\xi^2}{4} bt - \frac{\pi}{4} \right) . \]

Combining (5.27) with (5.26) yields

\[ y(x,t) = \frac{1}{2\sqrt{2\pi bt}} \int_{-\infty}^{\infty} f(x-\xi) \left[ \cos \frac{\xi^2}{4} bt + \sin \frac{\xi^2}{4} bt \right] d\xi \]

(5.28)

\[ = \frac{1}{2\sqrt{\pi bt}} \int_{-\infty}^{\infty} f(x-\xi) \cos \left( \frac{\xi^2}{4} bt - \frac{\pi}{4} \right) d\xi . \]

For the particular case of

\[ y(x,0) = f(x) = f_0 \exp\left(-\frac{x^2}{4a^2}\right) , \]  

(5.29)

where \( f_0 \) and \( a \) are constants, the solution becomes

\[ y(x,t) = \frac{f_0 \exp\left[-\frac{x^2 a^2}{4(a^2 + b^2 t^2)}\right]}{(1 + \frac{b^2 t^2}{a^2})^{1/4}} \cos \left[ -\frac{btx^2}{4(a^2 + b^2 t^2)} - \frac{1}{2} \tan^{-1}\left(\frac{bt}{a^2}\right) \right] . \]  

(5.30)
Solution (5.30) is shown graphically in Figure 5.4 for successive instants of time. This particularly interesting plot, taken from Morse [5.2], compares the beam motion (solid line) with that of the flexible string (dotted line). The constant $b$ was chosen so that the average wave velocity for the beam case was approximately equal to the phase velocity for the string motion. The figure clearly demonstrates the dispersive nature of the beam wave motion when compared with the nondispersive motion of the flexible string.
5.5 Wave Effects - Geometric Dispersion

It should be clear that the wave type solutions \( f(x + ct) \) or \( g(x - ct) \) will not satisfy the Euler-Bernoulli beam equation (5.6). Hence a disturbance of arbitrary form cannot be propagated at a constant speed. Therefore an arbitrary wave form must move with a variable speed and will then suffer a change in shape or be dispersed, see Figure 5.4.

To investigate this dispersion phenomenon further, let us try the simple harmonic wave solution

\[
y(x,t) = A \cos[k(x \pm ct)]
\]  \hspace{1cm} (5.31)

in equation (5.6). After cancelling the common factors one finds the relation

\[
2k^2 - c^2 = 0
\]

or

\[
c = kb = \frac{2\pi b}{\lambda} \hspace{1cm} (5.32)
\]

From (5.32) it is seen that the phase velocity is no longer constant but depends on the wave length \( \lambda \). Of course we could also express the phase velocity in terms of the frequency. This phenomenon is called geometric dispersion. It is a product of the geometric nature of the waveguide (i.e., the beam). We can therefore conclude that harmonic wavetrains with different frequencies propagate with different velocities. Consequently since an arbitrary wave form can be expressed as a linear combination of harmonic waves (think of Fourier series), one can then see why a wave

\* The term geometric is used to distinguish this behavior from material dispersion found say, in nonelastic media.
undergoing dispersion will change shape as it advances. Further information on the general nature of dispersion may be found in Leibovich and Seebass [5.9].

Recalling our discussion in Section 1.11, we found that the superposition of two harmonic waves of slightly different frequency produced a modulated result with waves moving in groups or packets, see Figure 1.11. The velocity of each packet, being the group velocity, \( c_g \), was given by (1.98) i.e.,

\[
\frac{d\omega}{dk} = \frac{c_g}{\lambda}
\]  

(1.98)

This result can also be expressed as

\[
c_g = c - \frac{d\omega}{d\lambda}
\]  

(5.33)

It turns out that the group velocity may be greater than, equal to, or less than the phase velocity. If \( c_g = c = \text{constant} \), then waves of all frequencies travel with the same speed and no dispersion will occur.

Interpreting the previous remarks within the context of our dispersive transverse waves on a beam we find that \( c_g = 2c = \frac{4\pi b}{\lambda} \). Hence the group velocity is twice the phase velocity for this case.

It is important to notice that the dispersion relation (5.32) predicts that waves of infinitely short wavelength travel with unlimited speed. This fact is contrary to physical reality and constitutes a basis for the rejection of Euler-Bernoulli beam theory for small wavelength (high frequency) wave motion. The reason why Euler-Bernoulli beam theory breaks down when the wavelength is small (comparable with the lateral dimensions of the beam) is that our original assumptions viii.) and viii.) are no longer valid. A little physical thought should convince the reader that small length waves will rotate and deform beam elements.
5.6 Rayleigh Beam Theory

As pointed out in the previous section, the elementary Euler-Bernoulli beam theory has some serious shortcomings for high frequency motion. We make here a refinement in this theory to account for the rotary motion of beam elements. This rotary correction was first applied by Lord Rayleigh, and hence we will use the terminology, Rayleigh beam theory.

The place where the rotary motion, i.e., the angular acceleration of beam elements, would be incorporated into the analysis lies in the moment equation (5.2). Taking this motion into account, (5.2) now would read

$$\frac{\partial M}{\partial x} = V - \rho I \frac{\partial^2 \theta}{\partial t^2},$$

(5.34)

where \( \theta \) is the angle of rotation of the beam element, and \( \rho \) and \( I \) as before being the material density and cross-sectional area moment of inertia, respectively.

For small deformations \( \theta = \frac{\partial y}{\partial x} \), and so (5.34) becomes

$$\frac{\partial M}{\partial x} = V - \rho I \frac{\partial^3 y}{\partial t^2 \partial x} \quad .$$

(5.35)

Combining (5.1) and (5.35) with (5.4) now gives

$$b^2 \frac{\partial^4 y}{\partial x^4} - \frac{a^2}{b^2} \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\partial^2 y}{\partial t^2} = 0 \quad ,$$

(5.36)

where we have taken the zero external loading case, \( b^2 = EI/\rho A \) and \( a^2 = I/A \). Equation (5.36) is then the governing relation for Rayleigh beam theory.

The separation of variables method may also be applied successfully to equation (5.36). Putting in the product form \( y = X(x)T(t) \), gives

$$X'''' + \frac{\gamma^2 a^2}{b^2} X'' - \frac{\gamma^2}{b^2} X = 0$$

(5.37)

$$T + \gamma^2 T = 0 \quad ,$$
where γ is again the separation constant.

The solution to \((5.37)_2\) is the same as for the Euler-Bernoulli case, see (5.12). The solution to \((5.37)_1\) is not quite so easy, but can be handled by putting in a solution form \(Ae^{rx}\). After cancelling common terms, this produces the characteristic equation

\[
r^4 + \frac{\gamma a^2}{b^2} r^2 - \frac{\gamma^2}{b^2} = 0. \tag{5.38}
\]

The roots of this equation are

\[
r_{1,2} = \pm \left\{ -\frac{\gamma a^2}{2b^2} - \left[\frac{\gamma a^4}{4b^4} + \frac{\gamma^2}{b^2}\right]^{1/2}\right\}^{1/2}
\]

\[
r_{3,4} = \pm \left\{ -\frac{\gamma a^2}{2b^2} + \left[\frac{\gamma a^4}{4b^4} + \frac{\gamma^2}{b^2}\right]^{1/2}\right\}^{1/2}.
\tag{5.39}
\]

So our solution may be written as

\[
X(x) = A_1e^{r_{1x}} + A_2e^{r_{2x}} + A_3e^{r_{3x}} + A_4e^{r_{4x}}, \tag{5.40}
\]

or by letting \(r_{1,2} = \pm im_1\) and \(r_{3,4} = \pm m_2\) we write

\[
X(x) = C_1 \sin m_1 x + C_2 \cos m_1 x
\]

\[
= C_3 \sinh m_2 x + C_4 \cosh m_2 x \quad . \tag{5.41}
\]

Notice that the Rayleigh solution form, (5.41) is quite similar to the Euler-Bernoulli result, (5.11). However, a detailed comparison of the frequencies and mode shapes would show a difference between the two theories.

Considering next the wave effects in the Rayleigh theory, we find by putting the simple harmonic wave form (5.31) into (5.36)

\[
c = \frac{2\pi b}{\sqrt{\lambda^2 + 4\pi^2 a^2}} \quad , \tag{5.42}
\]

and by (5.33) the group velocity turns out to be
\[ c_g = c + c^3 \left( \frac{\lambda}{2\pi b} \right)^2 \]  

(5.43)

So for the Rayleigh beam theory we find the physically more satisfying result that both the phase and group velocities remain bounded as the wavelength goes to zero.

5.7 **TIMOSHENKO BEAM THEORY**

The previous section corrected the elementary Euler-Bernoulli theory to account for rotary motion. We now further refine the theory to also account for shear deformation. Timoshenko in developing this theory has shown that shear deformation is at least as important as rotary inertia for high frequency vibrations. Hence following his work we will retain both rotary and shear effects.

We direct our attention to the shear deformation situation. For this case beam elements have cross-sectional faces which are no longer perpendicular to the neutral axis or lateral sides. This situation is shown in Figure 5.5, where \( \psi \) represents the angle the cross-section makes
with the $y$-axis. The shear strain at the cross-section is then $\frac{\partial y}{\partial x} - \psi$.

The moment and shear force are now given by

\begin{equation}
M = -EI \frac{\partial \psi}{\partial x},
\end{equation}

\begin{equation}
V = k'\mu A \left( \frac{\partial y}{\partial x} - \psi \right),
\end{equation}

where $\mu$ is the shear modulus and $k'$ is a shear constant depending on the shape of the cross-section. The moment equilibrium equation from the preceeding analysis (5.34) now becomes

\begin{equation}
EI \frac{\partial^2 \psi}{\partial x^2} + k'\mu A \left( \frac{\partial y}{\partial x} - \psi \right) - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0.
\end{equation}

For vertical translational equilibrium, equation (5.1) is still valid, and so for our case with zero distributed loading this equation reads

\begin{equation}
k'\mu \left( \frac{\partial^2 y}{\partial x^2} - \frac{\partial \psi}{\partial x} \right) - \rho \frac{\partial^2 y}{\partial t^2} = 0.
\end{equation}

Eliminating the variable $\psi$ from (5.45) and (5.46) produces the required result

\begin{equation}
b^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - a^2(1 + \frac{b^2 d^2}{a^2}) \frac{\partial^2 y}{\partial t^2 \partial x^2} + a^2 d^2 \frac{\partial^4 y}{\partial t^4} = 0,
\end{equation}

where $b^2 = EI/\rho A$, $a^2 = I/A$ and $d^2 = \frac{\rho}{k'\mu}$. Relation (5.47) is the Timoshenko beam theory equation and includes both rotary and shear corrections. Note in this expression that the first two terms correspond to the elementary case; the third term contains the Rayleigh correction for rotary inertia and a shear correction term; finally the last quantity is a coupling term between rotary and shear corrections.

To investigate the wave effects for this case, we again put form (5.31) into (5.47) which produces the following dispersion relation

\begin{equation}
c^4 - \frac{1}{a^2 d^2} \left( a^2 + b^2 d^2 + \frac{1}{k^2} \right) c^2 + \frac{b^2}{a^2 d^2} = 0.
\end{equation}
Although four roots exist for equation (5.48) only two are physically meaningful. These two roots typically would give a phase velocity dependence as shown in Figure 5.6. Each curve in the figure approaches the particular asymptotic limit as $k \to \infty$. The group velocity corresponding to these two phase velocity curves is shown in Figure 5.7. From the figure we see that the group velocity also approaches these asymptotic values for large $k$. The details of justifying these curves are saved for the exercises.

5.8 **Comparison of Euler-Bernoulli, Rayleigh, and Timoshenko Theories with Exact Elasticity Solution**

Having presented three strength of materials beam theories: Euler-Bernoulli, Rayleigh and Timoshenko, let us compare these theories with what is normally called the **exact theory**. It should be pointed out
that this exact theory is not the absolute perfection in beam theory but is an accepted standard by which approximate theories are judged.

The exact theory comes from the elasticity solution to the problem and is normally called the Pochhammer-Chree solution. We will discuss the general theory of elasticity and develop the Pochhammer-Chree solution in later chapters. For now let us accept this solution realizing that it is valid subject to linear elastic media undergoing small deformations.

For comparison purposes we consider the special case of flexural vibrations of a cylindrical bar of radius \( R \) with the elastic constant, Poisson's ratio, equalling .29. Using the results of Davies [5.10], Figures 5.8 and 5.9 illustrate the phase and group velocities for this case as a function of wave number. In each of the figures one can see the large deviation by the Euler-Bernoulli and Rayleigh theories from the first mode exact theory. It is only for very small values of wave number (large wave length) that these theories give accurate results.
On the other hand the Timoshenko theory agrees very well with the exact theory for both phase and group velocity predictions. Notice from Figure 5.9 that according to the exact and Timoshenko theories the group velocity exhibits a maximum value at about $\lambda = 3R$. Consequently for an arbitrary flexural wave, Fourier components having wavelengths about three times the radius of the bar will travel ahead of the other components, and hence will appear in front of the pulse.
Figure 5.9
REFERENCES: CHAPTER 5


