CHAPTER 10

FUNDAMENTAL EQUATIONS OF DYNAMIC ELASTICITY

10.1 Introduction

This chapter begins a more general study of wave motion in linear elastic continuous media. We have already discussed in earlier chapters wave motion in elastic rods, beams and plates. However, these previous analyses were made from the less rigorous strength of materials viewpoint, which makes certain assumptions about the deformation and stresses within a body. The theory to be introduced in this chapter makes much fewer assumptions, and attempts to attack problems from a more fundamental basis.

The *dynamic theory of elasticity*, sometimes called *elastodynamics**, formulates the propagation of mechanical disturbances by considering:

i.) the exact *kinematics of small deformation theory*,

ii.) the *conservation of linear and angular momentum*,

iii.) the general three-dimensional *material* or *constitutive behavior* of linear elastic media.

We now proceed to discuss these concepts. No attempt will be made to cover the entire field of elasticity theory. Rather we present the major aspects of the theory needed for a basic understanding and to pursue the

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* Elasticity theory can be separated into static and dynamic segments; the static part being called *elastostatics*.

study in the upcoming chapters. Further reading on the subject may be found in Sadd [10.1], Sokolnikoff [10.2], Timoshenko and Goodier [10.3], Little [10.4], Fung [10.5], or Chou and Pagano [10.6]. In addition to these sources (which emphasize elastostatics), the specific study of elastodynamics may also be found in Achenbach [10.7], Reismann and Pawlik [10.8], Clark [10.9], Kolsky [10.10], and Nowacki [10.11].

In this chapter all work will be done in rectangular Cartesian coordinate systems. Use of non-Cartesian systems will be considered in later chapters. Both *scalar* and *vector notation* will be emphasized, with some use of *index, i.e., Cartesian tensor, notation* in certain places.

10.2 Strain-Displacement Relations

We begin our study by looking at the kinematics of small deformation theory. Denote the *displacement* of a particle from the *undeformed configuration* to the *deformed configuration* by the vector $\underline{u}$ which has components u, v and w in a Cartesian coordinate system. Hence we write

$$\underline{u} = \{u,v,w\} \quad . \tag{10.1}$$

In index notation the vector $\underline{u}$ is written as u_i, where the subscript i ranges over the values 1,2,3. For this case we write

$$u_i = \{u_1,u_2,u_3\} \quad . \tag{10.2}$$

The correspondence $u = u_1$, $v = u_2$ and $w = u_3$ should be apparent.

A body is said to be *strained* or *deformed* when the *relative* positions of points in the body are changed. It should be noted that this definition distinguishes strain from *rigid body motion*. It is possible to define *three normal* or *longitudinal strain components* ε_x, ε_y and ε_z which are the *change in length per unit length* of infinitesimal fibers initially

lying along the x, y and z axes, respectively. It is further possible to define *six shearing strain components* γ_{xy}, γ_{yx}, γ_{yz}, γ_{zy}, γ_{zx} and γ_{xz} which represent *angle changes* of fibers initially along the x, y and z axes.

Hence there are *nine* strain components which make up the *strain matrix* or *strain tensor;* we write

$$\underset{\sim}{\varepsilon} = \varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix} \quad . \tag{10.3}$$

With regard to the indexed symbol ε_{ij}, it should be clear that $\varepsilon_{11} = \varepsilon_x$, $\varepsilon_{12} = \gamma_{xy}$, $\varepsilon_{13} = \gamma_{xz}$, $\varepsilon_{21} = \gamma_{yx}$, ...

By considering the change in shape of an infinitesimal element, the strains and displacements are found to be related by the following expressions

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{zy} \\ \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \gamma_{xz} \quad . \end{aligned} \tag{10.4}$$

These so-called *strain-displacement* relations then give the strain components in terms of the displacement gradients. Equations (10.4) may also be written in index notation as

$$e_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \quad , \tag{10.5}$$

where the new strain tensor is defined by

$$e_{ij} = \begin{cases} \varepsilon_{ij} & , \quad i = j \\ \frac{1}{2}\,\varepsilon_{ij} & , \quad i \neq j \end{cases} \quad , \tag{10.6}$$

and the *comma notation* means $(\ )_{,i} = \frac{\partial(\ )}{\partial x_i}$. Again one should see that $u_{1,1} = \frac{\partial u}{\partial x}$, $u_{1,2} = \frac{\partial u}{\partial y}$, ... Because $\gamma_{xy} = \gamma_{yx}$, $\gamma_{yz} = \gamma_{zy}$ and $\gamma_{zx} = \gamma_{xz}$, we can write $\varepsilon_{ij} = \varepsilon_{ji}$ or $e_{ij} = e_{ji}$, and then say that the strain matrix or tensor is *symmetric*.

The *angular rotation* $\underline{\omega}$ of an infinitesimal element will also be important to us. It can be shown that the displacement gradients are also related to this rotation by

$$\begin{aligned} \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned} \tag{10.7}$$

where $\underline{\omega} = \{\omega_x, \omega_y, \omega_z\}$. Note that (10.7) can be written in vector form as the cross-product

$$\underline{\omega} = \frac{1}{2}\,\underline{\nabla}\mathrm{x}\underline{u} = \frac{1}{2}\,\mathrm{curl}\,\underline{u} \quad . \tag{10.8}$$

Specifying the displacements, relations (10.4) will then give the strain components. Turning the problem around produces a dilemma. Because of the nature of equations (10.4), all six strain components cannot be arbitrarily prescribed if we are to have continuous single-valued displacements. Additional theory involving *strain compatibility* would be required, but having no need for this theory we omit its discussion.

10.3 FORCES AND STRESS

In general forces on a body may be grouped into two categories:

1. *Body Forces.* Forces proportional to the mass of the body, e.g., gravitational, magnetic, inertial, etc.
2. *Surfaces Forces.* Forces acting on the surface of a body, resulting from physical contact with another body.

Whether they be static or dynamic, applied external loads induce internal forces and stresses inside a body. *Stress* is defined as the *contact force per unit area* on an infinitesimal element. In three-dimensions since there are three force components and three directions for defining an areal element, we can define *nine* components of stress. Referring to Figure 10.1 we define *three normal stress components* σ_x, σ_y

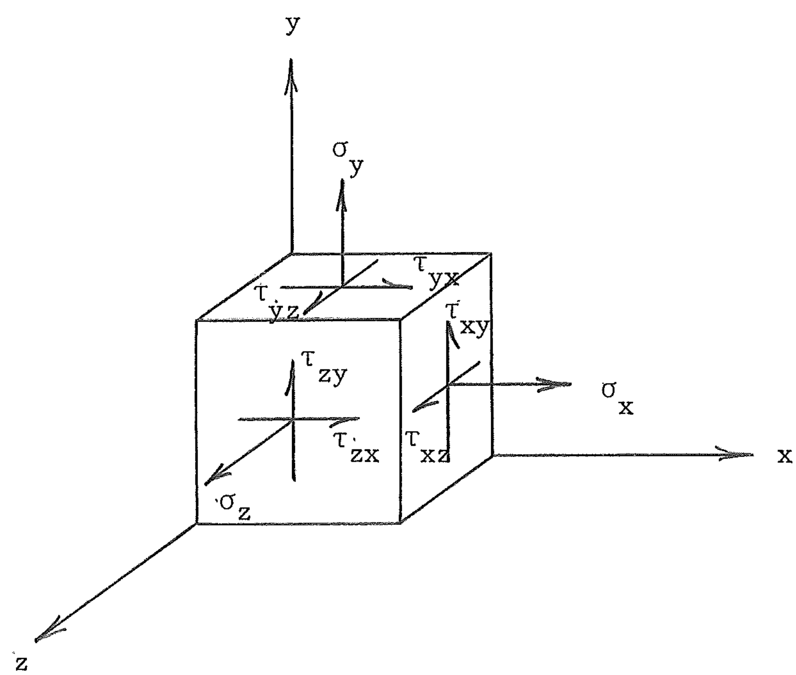

Figure 10.1

and σ_z as the normal force per unit area on the x, y and z faces of the element. In addition *six shearing stress components* are defined by τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz} which represent the various shear force per unit

area ratios.

We then can write the *stress matrix* or *stress tensor* as

$$\underset{\sim}{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

One can define a *stress* or *traction vector* $\underline{T}^n$ by the relation

$$\underline{T}^n = \underset{\sim}{\sigma} \cdot \underline{n} \quad , \tag{10.10}$$

where the dot product is interpreted as

$$\begin{aligned} T_x^{\,n} &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ T_y^{\,n} &= \tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z \\ T_z^{\,n} &= \tau_{zx} n_x + \tau_{zy} n_y + \sigma_z n_z \quad . \end{aligned} \tag{10.11}$$

The tötal force acting on an area A, is then given by $\int_A \underline{T}^n \cdot \underline{n} dA$, where $\underline{n}$ is the unit normal vector to the surface.

10.4 EQUATIONS OF MOTION

Having discussed the forces and stresses in a continuum, we now apply two basic *balance principles*, the *conservation of linear and angular momentum*. Rather than being completely general by considering an arbitrary shaped element, we follow the more instructive approach by taking the rectangular two-dimensional element shown in Figure 10.2. In the figure F_x and F_y represent the body force per unit volume in the x and y directions.

The conservation of linear momentum, which for this case is actually Newton's second law, gives the following two equations in the x and y directions

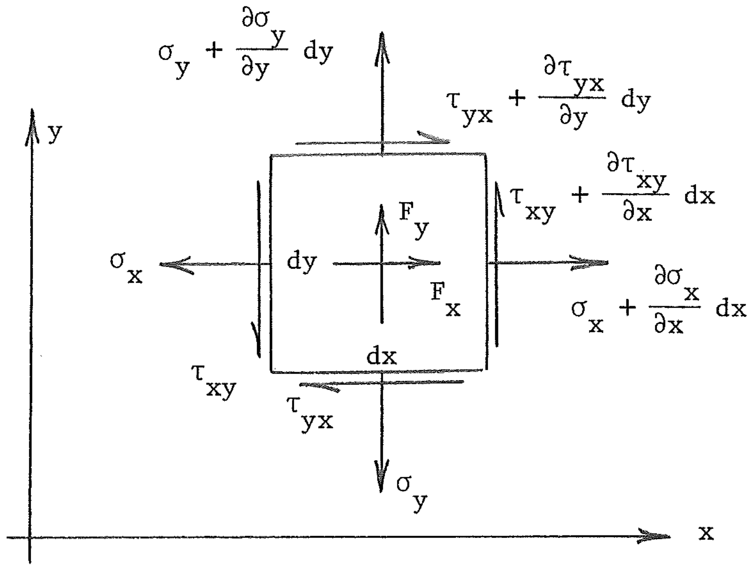

Figure 10.2

$$
\begin{aligned}
&(\sigma_x + \frac{\partial\sigma_x}{\partial x}dx)dy - \sigma_x dy + (\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}dy)dx \\
&\quad - \tau_{yx}dx + F_x dxdy = \rho dxdy \frac{\partial^2 u}{\partial t^2} \\
&(\sigma_y + \frac{\partial\sigma_y}{\partial y}dy)dx - \sigma_y dx + (\tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}dx)dy \\
&\quad - \tau_{xy}dy + F_y dxdy = \rho dxdy \frac{\partial^2 v}{\partial t^2}
\end{aligned}
\tag{10.12}
$$

where ρ is the mass density per unit volume. Dividing (10.12) through by dxdy and letting dx and dy go to zero gives

$$
\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + F_x &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + F_y &= \rho \frac{\partial^2 v}{\partial t^2} \quad .
\end{aligned}
\tag{10.13}
$$

By following exactly the same procedure, in three dimensions one would get

$$
\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + F_x &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} + F_y &= \rho \frac{\partial^2 v}{\partial t^2} \\
\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + F_z &= \rho \frac{\partial^2 w}{\partial t^2} \quad .
\end{aligned}
\tag{10.14}
$$

In index notation, equations (10.14) may be written as

$$\sigma_{ji,j} + F_i = \rho\ddot{u}_i \quad , \tag{10.15}$$

where repeated indices imply summation over the range 1, 2, 3, i.e., $\sigma_{ji,j} = \sum_{j=1}^{3} \frac{\partial\sigma_{ji}}{\partial x_j}$. Relations (10.14) are normally called the *equations of motion*, and are valid for all continua. With zero acceleration terms, (10.14) become the *equilibrium equations*.

To apply the conservation of angular momentum or moment of momentum, we sum moments about a fixed point in Figure 10.2. The resulting expression in the limit as dx and dy go to zero gives simply

$$\tau_{xy} = \tau_{yx} \quad . \tag{10.16}$$

Again in three dimensions one can show that

$$\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{yz} &= \tau_{zy} \\ \tau_{zx} &= \tau_{xz} \quad . \end{aligned} \tag{10.17}$$

Note that this result implies that the stress matrix or tensor is symmetric, i.e., $\sigma_{ij} = \sigma_{ji}$.

10.5 Linear Elastic Constitutive Relation: Hooke's Law

Reviewing what we have done so far, the strain-displacement relations (10.4) and the equations of motion (10.14) are valid for all continua undergoing small deformations. In addition, these two sets of equations represent 9 equations for a total of 15 unknowns (6 independent strains, 6 independent stresses, and 3 displacements). Therefore we need 6 more relations which of course must bring the *material response* into the

problem. Consideration will now be directed to these equations which relate the stress (force) to the strain (deformation) and are normally called *constitutive relations*.

For the case of linear elastic media, we postulate that each stress component is linearly related to each strain component, i.e.,

$$
\begin{aligned}
\sigma_x &= C_{11}\varepsilon_x + C_{12}\varepsilon_y + C_{13}\varepsilon_z + C_{14}\gamma_{xy} + C_{15}\gamma_{yz} + C_{16}\gamma_{zx} \\
\sigma_y &= C_{21}\varepsilon_x + C_{22}\varepsilon_y + \ldots \\
\sigma_z &= C_{31}\varepsilon_x + \ldots \\
\tau_{xy} &= C_{41}\varepsilon_x + \ldots \\
\tau_{yz} &= C_{51}\varepsilon_x + \ldots \\
\tau_{zx} &= C_{61}\varepsilon_x + \ldots \quad ,
\end{aligned}
\tag{10.18}
$$

or in index notation

$$
\sigma_{ij} = C_{ijk\ell}e_{k\ell} \quad . \tag{10.19}
$$

Relation (10.18) or (10.19) is called the *generalized Hooke's Law*. If the media is taken to be *homogeneous*, then the coefficients C_{11}, C_{12}, ... are independent of the space coordinates and only depend on the material. If the media is also *isotropic* (i.e., has no preferred directions), then the general form (10.18) reduces to

$$
\begin{aligned}
\sigma_x &= \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_x \\
\sigma_y &= \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_y \\
\sigma_z &= \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_z \\
\tau_{xy} &= \mu\gamma_{xy} \\
\tau_{yz} &= \mu\gamma_{yz} \\
\tau_{zx} &= \mu\gamma_{zx} \quad ,
\end{aligned}
\tag{10.20}
$$

where λ and μ are material constants being, respectively, *Lame's constant*

and the *shear modulus*. Relations (10.20) written in index notation read

$$\sigma_{ij} = \lambda \vartheta \delta_{ij} + 2\mu e_{ij} \quad , \tag{10.21}$$

where ϑ is the *dilatation* given by

$$\vartheta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_{kk} \quad , \tag{10.22}$$

and δ_{ij} is the *Kronecker delta* defined as

$$\delta_{ij} = \begin{cases} 1 \; , & i = j \\ 0 \; , & i \neq j \end{cases} \quad . \tag{10.23}$$

It can be shown that the dilatation is equal to the change in volume per unit volume.

Equations (10.20) or (10.21) may be inverted to get the strain components in terms of the stresses. The result is

$$\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \\
\varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) \\
\varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \\
\gamma_{xy} &= \frac{1}{\mu}\tau_{xy} \\
\gamma_{yz} &= \frac{1}{\mu}\tau_{yz} \\
\gamma_{zx} &= \frac{1}{\mu}\tau_{zx} \quad ,
\end{aligned} \tag{10.24}$$

or in index notation

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\theta \quad , \tag{10.25}$$

where

$$\theta = \sigma_x + \sigma_y + \sigma_z = \sigma_{kk} \quad , \tag{10.26}$$

and the two new material constants *Poisson's ratio* ν, and the *modulus of elasticity* or *Young's modulus* E, are given by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$
$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad . \tag{10.27}$$

Considering the case of *hydrostatic pressure*

$$\sigma_x = \sigma_y = \sigma_z = -p$$
$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \quad , \tag{10.28}$$

we find from (10.24) that

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -\left(\frac{1-2\nu}{E}\right) p$$
$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \quad . \tag{10.29}$$

So the dilatation is

$$\vartheta = -\frac{3}{E}(1-2\nu)\, p \quad , \tag{10.30}$$

and defining a *bulk modulus of elasticity* k, by

$$k = -\frac{p}{\vartheta} \quad , \tag{10.31}$$

we find

$$k = \frac{E}{3(1-2\nu)} \quad . \tag{10.32}$$

In the discussion so far we have defined *five* elastic constants λ, μ, E, ν and k. However, for isotropic media they are all interrelated so that only *two* of them are independent. Table 10.1 gives the relations between any three elastic constants. From this table any one elastic constant can be expressed in terms of any two other constants, e.g.,

relations (10.27) or (10.32).

Table 10.1

	E	$\sigma = \nu$	$\kappa = k$	μ	λ
E, σ	E	σ	$\frac{E}{3(1-2\sigma)}$	$\frac{E}{2(1+\sigma)}$	$\frac{E\sigma}{(1+\sigma)(1-2\sigma)}$
E, κ	E	$\frac{3\kappa-E}{6\kappa}$	κ	$\frac{3\kappa E}{9\kappa-E}$	$\frac{3\kappa(3\kappa-E)}{9\kappa-E}$
E, μ	E	$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$	μ	$\frac{\mu(E-2\mu)}{3\mu-E}$
E, λ*	E	$\frac{2\lambda}{E+\lambda+\surd}$	$\frac{E+3\lambda+\surd}{6}$	$\frac{E-3\lambda+\surd}{4}$	λ
σ, κ	$3\kappa(1-2\sigma)$	σ	κ	$\frac{3\kappa(1-2\sigma)}{2(1+\sigma)}$	$\frac{3\kappa\sigma}{1+\sigma}$
σ, μ	$2\mu(1+\sigma)$	σ	$\frac{2\mu(1+\sigma)}{3(1-2\sigma)}$	μ	$\frac{2\mu\sigma}{1-2\sigma}$
σ, λ	$\frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma}$	σ	$\frac{\lambda(1+\sigma)}{3\sigma}$	$\frac{\lambda(1-2\sigma)}{2\sigma}$	λ
κ, μ	$\frac{9\kappa\mu}{6\kappa+\mu}$	$\frac{3\kappa-2\mu}{6\kappa+2\mu}$	κ	μ	$\kappa-\frac{2}{3}\mu$
κ, λ	$\frac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda}$	$\frac{\lambda}{3\kappa-\lambda}$	κ	$\frac{3}{2}(\kappa-\lambda)$	λ
μ, λ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2\lambda+2\mu}$	$\frac{3\lambda+2\mu}{3}$	μ	λ

* $\surd$ stands for $\sqrt{(E^2 + 9\lambda^2 + 2E\lambda)}$.

10.6 Equations of Motion in Terms of Displacements

By using Hooke's law (10.20) and the strain displacement relations (10.4), one can eliminate the stresses in the equations of motion (10.14). This produces the equations of motion in terms of the *displacements*. Saving the details for the exercises, the results of this process for zero body forces are

$$
\begin{aligned}
\mu\nabla^2 u + (\lambda + \mu)\frac{\partial\vartheta}{\partial x} &= \rho\frac{\partial^2 u}{\partial t^2} \\
\mu\nabla^2 v + (\lambda + \mu)\frac{\partial\vartheta}{\partial y} &= \rho\frac{\partial^2 v}{\partial t^2} \qquad (10.33) \\
\mu\nabla^2 w + (\lambda + \mu)\frac{\partial\vartheta}{\partial z} &= \rho\frac{\partial^2 w}{\partial t^2} \quad ,
\end{aligned}
$$

where now the dilatation is written as

$$\vartheta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad . \tag{10.34}$$

These results can also be written in vector notation as

$$\mu\nabla^2\underline{u} + (\lambda + \mu)\nabla(\underline{\nabla} \cdot \underline{u}) = \rho\ddot{\underline{u}} \quad , \tag{10.35}$$

while in index notation the equations read

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} = \rho\ddot{u}_i \quad . \tag{10.36}$$

As a review we point out that equations (10.33) are valid for isotropic homogeneous linear elastic media undergoing small deformations.

10.7 Dilatational and Equivoluminal Wave Motion

Consider first the case of wave motion propagating changes in volume, which is associated with the dilatation, ϑ. Taking the divergence of equation (10.35) gives

$$\mu\nabla^2(\underline{\nabla} \cdot \underline{u}) + (\lambda + \mu)\nabla^2(\underline{\nabla} \cdot \underline{u}) = \rho(\ddot{\overline{\underline{\nabla} \cdot \underline{u}}})$$

or

$$(\lambda + 2\mu)\nabla^2\vartheta = \rho\ddot{\vartheta} \quad , \tag{10.37}$$

where we have used (10.34). Rewriting (10.37) in standard form produces

$$\nabla^2\vartheta = \frac{1}{c_1^2}\ddot{\vartheta} \quad , \tag{10.38}$$

where $c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$, and is the velocity of this *dilatational wave motion*.

Note for *irrotational motion*

$$\underline{\nabla x u} = \text{curl}\ \underline{u} = 0 \quad , \tag{10.39}$$

and this implies, at least for *simply connected regions**, that

$$\underline{u} = \underline{\nabla}\phi \quad , \tag{10.40}$$

where ϕ is some arbitrary potential function. Using (10.40) in (10.35) gives

$$(\lambda + 2\mu)\nabla^2(\underline{\nabla}\phi) = \rho\ddot{\overline{(\underline{\nabla}\phi)}} \quad , \tag{10.41}$$

which is the same form as (10.37). Hence we have discovered the fact that *dilatational wave motion coincides with irrotational wave motion.*

Next consider the situation of *equivoluminal* (volume preserving) *wave motion.* For this case the dilatation is zero, i.e., $\vartheta = \underline{\nabla} \cdot \underline{u} = 0$, and so (10.35) gives

$$\mu\nabla^2\underline{u} = \rho\underline{\ddot{u}} \tag{10.42}$$

or

$$\nabla^2\underline{u} = \frac{1}{c_2^2}\underline{\ddot{u}} \tag{10.43}$$

where $c_2 = \sqrt{\frac{\mu}{\rho}}$, and is the velocity of the equivoluminal wave motion.

For *rotational* or *shearing motion* we recall our definition of the rotation vector $\underline{\omega}$ given in equations (10.7) and (10.8). Taking the *curl* (i.e., $\underline{\nabla}\mathrm{x}\cdot$) of equation (10.35) produces

$$\mu\nabla^2(\underline{\nabla}\mathrm{x}\underline{u}) = \rho\ddot{\overline{(\underline{\nabla}\mathrm{x}\underline{u})}} \quad ,$$

which can be written as

$$\mu\nabla^2\underline{\omega} = \rho\underline{\ddot{\omega}} \quad . \tag{10.44}$$

---

* A *simply connected region* is one in which any arbitrary closed contour can be continuously reduced to a point without going outside the region.

Again noting the similarity between equations (10.42) and (10.44) we conclude that *equivoluminal wave motion coincides with rotational or shear wave motion.* The terminology *distortional wave* is also used for this type of disturbance.

It should be clear that an arbitrary initial input disturbance into an elastic body will produce *both* dilatational and shear motions. However since these motions propagate with different velocities, dilatational and shear effects will separate from one another. Each motion will then propagate through the media acting independent of the other. The question does arise however; are these two types of motion the only ones possible in unbounded elastic media? The answer to this question will be discussed in a later section.

We conclude this section with Table 10-2, taken from Kolsky [10.10], giving some approximate numerical values of the wave velocities c_1 and c_2 (along with other wave speeds) for some common materials.

Table 10-2

	Steel	*Copper*	*Aluminium*	*Glass*	*Rubber*
Elastic constants (dynes/sq. cm.)					
λ	$11{\cdot}2 \times 10^{11}$	$9{\cdot}5 \times 10^{11}$	$5{\cdot}6 \times 10^{11}$	$2{\cdot}8 \times 10^{11}$	$1{\cdot}0 \times 10^{10}$
μ	8·1 ,,	4·5 ,,	2·6 ,,	2·8 ,,	$7{\cdot}0 \times 10^{6}$
E	21·0 ,,	12·0 ,,	7·0 ,,	7·0 ,,	$2{\cdot}0 \times 10^{7}$
k	16·7 ,,	12·5 ,,	7·3 ,,	4·7 ,,	$1{\cdot}0 \times 10^{10}$
Poisson's ratio ν	0·29	0·34	0·34	0·25	0·5
Density ρ	7·8	8·9	2·7	2·5	0·93
Velocities (metres/sec.)					
c_1	5,940	4,560	6,320	5,800	1,040
c_2	3,220	2,250	3,100	3,350	27
c_0	5,190	3,670	5,090	5,300	46
c_s	2,980	2,120	2,920	3,080	26
c_L	5,420	3,900	5,410	5,460	53

Note defining κ as the ratio

$$\kappa = \frac{c_1}{c_2} = \sqrt{\frac{2-2\nu}{1-2\nu}} \quad , \tag{10.45}$$

we see that for real materials, $0 < \nu < \frac{1}{2}$, and so $\sqrt{2} < \kappa < \infty$ which implies $c_1 > c_2$. Hence the dilatational wave speed is always *greater* than the shear wave speed.

10.8 Amplitude Behavior for Plane Dilatational and Shear Waves

In order to gain a better understanding of the behavior of dilatational and shear waves, let us consider the case of propagation of a plane wave represented by the displacement

$$\underline{u} = \underline{A}\, f(\psi) \quad , \tag{10.46}$$

where $\underline{A}$ is a constant amplitude vector and $\psi = \underline{r} \cdot \underline{n} - ct = \ell x + my + nz - ct$.

From the form (10.46) it follows that

$$\begin{aligned} \nabla^2\underline{u} &= \underline{A}f'' \\ \underline{\nabla} \cdot \underline{u} &= \underline{A} \cdot \underline{n}f' \\ \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) &= (\underline{A} \cdot \underline{n})\underline{n}f'' \\ \ddot{\underline{u}} &= c^2\underline{A}f'' \quad . \end{aligned} \tag{10.47}$$

Substituting (10.47) into the equations of motion (10.35) gives

$$\{\mu\underline{A} + (\lambda + \mu)(\underline{A} \cdot \underline{n})\underline{n} - \rho c^2\underline{A}\}f'' = 0 \quad . \tag{10.48}$$

We rule out the case $f'' = 0$; therefore

$$\mu\underline{A} + (\lambda + \mu)(\underline{A} \cdot \underline{n})\underline{n} - \rho c^2\underline{A} = 0 \quad . \tag{10.49}$$

Now we first consider the case of $\underline{A}$ *parallel* to $\underline{n}$. Taking the dot product of (10.49) with $\underline{n}$ gives

$$\{(\lambda + 2\mu) - \rho c^2\}(\underline{A} \cdot \underline{n}) = 0 \quad ,$$

and so since $\underline{A} \cdot \underline{n} \neq 0$ we find

$$c = c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad .$$

Likewise considering the case of $\underline{A}$ *perpendicular* to $\underline{n}$, then $\underline{A} \cdot \underline{n} = 0$, and so taking the dot product of (10.49) with $\underline{A}$ produces

$$\{\mu - \rho c^2\}(\underline{A} \cdot \underline{A}) = 0 \quad .$$

Since $\underline{A} \cdot \underline{A} = A^2 \neq 0$, then

$$c = c_2 = \sqrt{\frac{\mu}{\rho}} \quad .$$

In review, we have shown for the plane wave case that for dilatational waves the amplitude is normal to the wave surface, while for shear waves the amplitude is in the plane of the wave front.

For this plane wave case consider next the traction vector defined in equation (10.10) or (10.11). By a similar analysis one can show that the traction vector on the wave front is

$$\underline{T}^n = \{\underline{A} + \frac{1}{1-2\nu}(\underline{A} \cdot \underline{n})\underline{n}\}\mu f' \quad . \tag{10.50}$$

It then follows that for the dilatational wave $\underline{A}$ is parallel to $\underline{n}$ and (10.50) implies that

$$\underline{T}^n = \frac{2-2\nu}{1-2\nu}\mu f'\underline{A} \quad , \tag{10.51}$$

and so no shearing stresses will be present. For the shear wave, $\underline{A}$ is perpendicular to $\underline{n}$ so (10.50) now gives

$$\underline{T}^n = \mu f'\underline{A} \quad , \tag{10.52}$$

and hence no normal stresses will act on the wave surface. It should be realized that these informative results are only true for plane waves.

10.9 Basic Boundary Value Problems and Uniqueness

Having presented the proper equations to determine the unknown stresses, strains and displacements, we can now pose the fundamental boundary value problems in linear elastodynamics. Equations (10.4), (10.14) and (10.20) compose the fundamental set of equations to be solved subject to certain boundary and initial conditions, and with given body forces. Normally most problems are formulated with specified tractions and/or displacements on the boundary. Three basic boundary value problems therefore emerge:

Problem 1: Traction Conditions. Determine the distribution of stress and displacement in the interior of an elastic body in dynamic equilibrium when the body forces are given and the distribution of *surface forces are prescribed on the surface of the body.*

Problem 2: Displacement Conditions. Determine the distribution of stress and displacement in the interior of an elastic body in dynamic equilibrium when the body forces are given and the *displacements of the points on the surface are prescribed.*

Mixed Problem: Determine the distribution of stress and displacement in the interior of an elastic body in dynamic equilibrium when the body forces are given and the *surface forces are prescribed over part of the surface* of the body while the *displacements are prescribed over the remaining portion of the surface.*

In all of these problem types *initial conditions* on the displacement and velocity throughout the body are also needed.

The question of *uniqueness* involving a solution to our fundamental set of equations is a theoretical concept worth caring about from a practical viewpoint if we are to have meaningful results. It is necessary to know that a solution found by some special ad hoc technique

is the only solution to the problem. The *Neumann Uniqueness Theorem* asserts that for finite-sized bodies the solutions to the traction, displacement or mixed problems *are unique*. We will not prove this theorem here; the interested reader is referred to Sokolnikoff [10.2], Achenbach [10.7] or Reismann and Pawlik [10.8]. Uniqueness has also been extended to infinite domains, see Wheeler and Sternberg [10.12].

10.10 HELMHOLTZ DECOMPOSITION THEOREM

A useful theorem in mathematics states that every *sufficiently smooth* vector field $\underline{u}$ may be written as the sum of the *gradient of a potential function* plus the *curl of a vector function*, i.e.,

$$\underline{u} = \underline{\nabla}\phi + \underline{\nabla}\times\underline{\psi} \quad . \tag{10.53}$$

with

$$\underline{\nabla} \cdot \underline{\psi} = 0 \quad . \tag{10.54}$$

The term *sufficiently smooth* means that the vector $\underline{u}$ must satisfy certain continuity requirements; see Fung [10.5] or Achenbach [10.7] for details. The representation (10.53) is normally called the *Helmholtz decomposition*.

Interpreting $\underline{u}$ in (10.53) as the elastodynamic displacement vector, let us substitute this form into the equations of motion (10.35). This produces

$$\underline{\nabla}[(\lambda + 2\mu)\nabla^2\phi - \rho\ddot{\phi}] + \underline{\nabla}\times[\mu\nabla^2\underline{\psi} - \rho\ddot{\underline{\psi}}] = 0 \quad . \tag{10.55}$$

It then follows that (10.55) will be satisfied if ϕ and $\underline{\psi}$ satisfy

$$\begin{aligned} \nabla^2\phi &= \frac{1}{c_1^2}\ddot{\phi} \\ \nabla^2\underline{\psi} &= \frac{1}{c_2^2}\ddot{\underline{\psi}} \quad . \end{aligned} \tag{10.56}$$

Consequently the ϕ disturbance is associated with the dilatational wave motion while $\underline{\psi}$ is connected with the shear wave propagation. As a matter of fact ϕ and $\underline{\psi}$ are related to the dilatation ϑ and rotation $\underline{\omega}$ by

$$\begin{aligned} \nabla^2\phi &= \vartheta \\ \nabla^2\underline{\psi} &= -2\underline{\omega} \quad . \end{aligned} \tag{10.57}$$

It is important to realize that the shear wave equations (10.43), (10.44) or $(10.56)_2$ are *vector* wave equations and actually represent three scalar equations.

Notice that by introducing the potentials ϕ and $\underline{\psi}$ we have reduced the general problem in elastodynamics to that of solving the familiar wave equation. And although the two potentials are generally coupled through the boundary conditions, use of this representation normally simplifies the analysis.

As mentioned, in general the boundary conditions will couple the two potentials ϕ and $\underline{\psi}$. For displacement boundary conditions this fact should be apparent directly from the original form (10.53). For the traction or stress boundary conditions one needs to first express the stresses in terms of the potentials. This produces the following set of equations

$$\begin{aligned} \sigma_x &= \lambda\nabla^2\phi + 2\mu\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z}\right)\right] \\ \sigma_y &= \lambda\nabla^2\phi + 2\mu\left[\frac{\partial^2\phi}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial\psi_x}{\partial z} - \frac{\partial\psi_z}{\partial x}\right)\right] \\ \sigma_z &= \lambda\nabla^2\phi + 2\mu\left[\frac{\partial^2\phi}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y}\right)\right] \\ \tau_{xy} &= \mu\left[2\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\psi_x}{\partial z} - \frac{\partial\psi_z}{\partial x}\right)\right] \end{aligned} \tag{10.58}$$

$$\tau_{yz} = \mu[2 \frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial}{\partial z} (\frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x}) + \frac{\partial}{\partial y} (\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y})]$$

$$\tau_{zx} = \mu[2 \frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial}{\partial x} (\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}) + \frac{\partial}{\partial z} (\frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z})] \quad ,$$

where $\underline{\psi} = \{\psi_x, \psi_y, \psi_z\}$.

It is then obvious that specification of certain stresses will result in an expression containing both ϕ and $\underline{\psi}$.

As a final note we consider briefly the question of *completeness of the Helmholtz decomposition.* As one might expect not all solutions of (10.55) are given by (10.56). However, one can show that every solution of the displacement equations of motion admits the representation (10.53); for details see Fung [10.5], Achenbach [10.7] or Sternberg [10.13]. This in a sense answers the question posed in Section 10.7, and says that dilatational and shear wave motion are the only types possible in unbounded elastic media.

10.11 TWO-DIMENSIONAL FORMULATION

Because of the deformation or stress field, many problems in elastodynamics may be modeled by a two-dimensional theory. This simplification will be very useful for problem solution. Basically there exist two different concepts, *plane strain* and *plane stress*, which yield two-dimensional formulations.

Plane Strain

A state of plane strain exists in a body when the *displacement components* take the form

$$u = u(x,y,t)$$
$$v = v(x,y,t) \qquad (10.59)$$
$$w = 0 \quad .$$

The strains and stresses corresponding to (10.59) follow from (10.4) and (10.20) to be

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\varepsilon_z &= \gamma_{yz} = \gamma_{zx} = 0 \quad ,
\end{aligned}
\tag{10.60}
$$

and

$$
\begin{aligned}
\sigma_x &= \lambda(\varepsilon_x + \varepsilon_y) + 2\mu\varepsilon_x \\
\sigma_y &= \lambda(\varepsilon_x + \varepsilon_y) + 2\mu\varepsilon_y \\
\sigma_z &= \lambda(\varepsilon_x + \varepsilon_y) = \nu(\sigma_x + \sigma_y) \\
\tau_{xy} &= \mu\gamma_{xy} \\
\tau_{yz} &= \tau_{zx} = 0 \quad .
\end{aligned}
\tag{10.61}
$$

The equations of motion become

$$
\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y &= \rho \frac{\partial^2 v}{\partial t^2} \quad ,
\end{aligned}
\tag{10.62}
$$

or in terms of displacements

$$
\mu\nabla^2\underline{u} + (\lambda + \mu)\underline{\nabla}(\underline{\nabla} \cdot \underline{u}) + \underline{F} = \rho\ddot{\underline{u}} \quad , \tag{10.63}
$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\underline{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j}$ and so

$$
\underline{\nabla} \cdot \underline{u} = \vartheta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad . \tag{10.64}
$$

Plane Stress

A state of plane stress is said to exist in a body when the *stress components* are of the form

$$
\begin{aligned}
\sigma_x &= \sigma_x(x,y,t) \\
\sigma_y &= \sigma_y(x,y,t) \\
\tau_{xy} &= \tau_{xy}(x,y,t) \\
\sigma_z &= \tau_{yz} = \tau_{zx} = 0 \quad .
\end{aligned}
\tag{10.65}
$$

The strain components follow from (10.24)

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\varepsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) \\
\gamma_{xy} &= \frac{1}{\mu}\tau_{xy} \\
\gamma_{yz} &= \gamma_{zx} = 0 \quad .
\end{aligned}
\tag{10.66}
$$

For this case the equations of motion in terms of the stresses are the same as for plane strain i.e., (10.62), while in terms of displacements the equations read

$$\mu\nabla^2\underline{u} + \frac{E}{2(1-\nu)}\ \underline{\nabla}(\underline{\nabla}\cdot\underline{u}) + \underline{F} = \rho\ddot{\underline{u}} \quad . \tag{10.67}$$

Note that (10.67) is not identical to the corresponding plane strain result (10.63). However, it can be shown that if we replace E by $\frac{E}{1-\nu^2}$ and ν by $\frac{\nu}{1-\nu}$ in the plane stress result (10.67), it will reduce to (10.63). Hence solutions of the two different theories are related by a simple change in the elastic constants E and ν.

In our development of this plane stress formulation we actually neglected some of the strain-displacement relations. These equations will not in general be satisfied by a solution to a plane stress problem, hence our solution is only an approximate one. It turns out that this approximation is very good for elastic bodies *thin* in the z-direction; whereas the plane strain approximation is more realistic for bodies very *thick* in the z-direction.

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