MCE 561 Computational Methods in Solid Mechanics
Time Dependent Problems

Time dependent problems occur in many areas of engineering science, e.g. transient diffusion (heat transfer), vibrations and wave propagation. In these types of problems the governing field equations contain *first and/or second order time derivatives*, see for example the transient diffusion and wave equation in the basic field equations class handout. For these types of problems we have both *spatial and temporal* (time) variations to approximate, and therefore this normally leads to two stages of solution. The first stage considers the spatial approximation (i.e. the finite element discretization), and this is then followed by a time approximation scheme to handle or generate a *time-stepping* method. Such procedures are commonly classified as *semidiscrete approximations* in the spatial sense, and these methods reduce the governing field equations to a set of *ordinary differential equations* in time. The final step to incorporate a time marching scheme ultimately involves the reduction of the problem to a set of *algebraic equations*.

Approximation of First-Order Time Derivatives

As mentioned for field equations involving first order time derivatives, the spatial discretization will typically produce the following matrix differential equation

$$[A]\{\dot{u}\} + [B]\{u\} = \{P\} \tag{1}$$

In order to develop a time-stepping scheme to handle the first order derivative, we introduce the so-called α - *family of approximation*. In this method the weighted time average of the time derivative of the dependent variable at two consecutive time steps is approximated by a linear interpolation of the values of the variable at the two steps, i.e.

$$\alpha\{\dot{u}\}_{n+1} + (1 - \alpha)\{\dot{u}\}_n = \frac{\{u\}_{n+1} - \{u\}_n}{\Delta t_{n+1}} \tag{2}$$

where $0 \le \alpha \le 1$, $\{\bullet\}_n$ denotes to the value at time t_n, and $\Delta t_n = t_n - t_{n-1}$. From the above method, several well-known difference schemes can be obtained by choosing various values of the α parameter, i.e.

$$\alpha = \begin{cases} 0 & \textit{Forward Difference, Euler - Conditionally Stable,} \\ \dfrac{1}{2} & \textit{Crank-Nicolson, - Stable, } O(\Delta t)^2 \\ \dfrac{2}{3} & \textit{Galerkin, - Stable, } O(\Delta t)^2 \\ 1 & \textit{Backward Difference Stable, } O(\Delta t) \end{cases} \tag{3}$$

Using the approximation (2) for times t_n and t_{n+1} in equation (1) yields

$$\begin{aligned} [A]\{u\}_{n+1} = [A]\{u\}_n &+ \alpha\Delta t_{n+1}(\{P\}_{n+1} - [B]\{u\}_{n+1}) \\ &+ (1 - \alpha)\Delta t_{n+1}(\{P\}_n - [B]\{u\}_n) \end{aligned} \tag{4}$$

Rearranging the terms to express $\{u\}_{n+1}$ in terms of $\{u\}_n$, yields

$$([A] + \alpha\Delta t_{n+1}[B])\{u\}_{n+1} = ([A] - (1 - \alpha)\Delta t_{n+1}[B])\{u\}_n + \Delta t_{n+1}(\alpha\{P\}_{n+1} + (1 - \alpha)\{P\}_n) \tag{5}$$

or

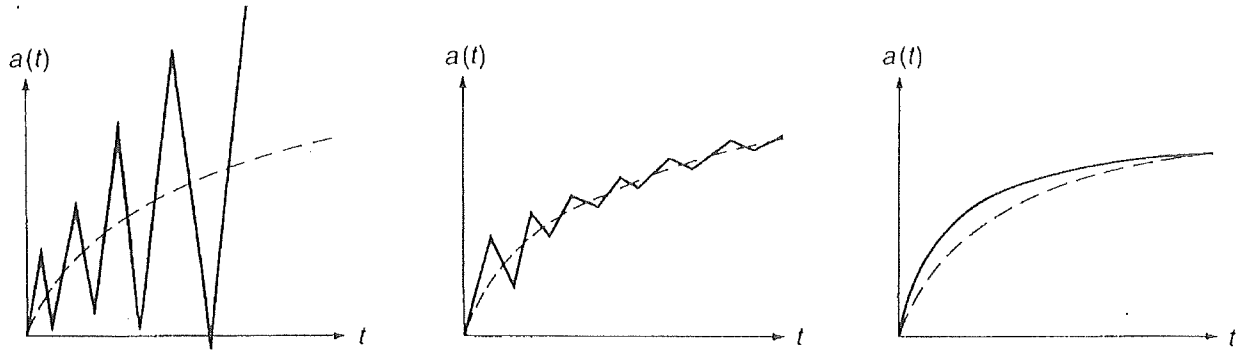
$$[\hat{A}]\{u\}_{n+1} = [\hat{B}]\{u\}_n + \{P\}_{n,\,n+1} \tag{6}$$

where

$$\begin{aligned} [\hat{A}] &= [A] + \alpha\Delta t_{n+1}[B] \\ [\hat{B}] &= [A] - (1-\alpha)\Delta t_{n+1}[B] \\ \{P\}_{n,\,n+1} &= \Delta t_{n+1}[\alpha\{P\}_{n+1} + (1-\alpha)\{P\}_n] \end{aligned} \tag{7}$$

Thus the solution at time t_{n+1} is obtained in terms of the values at time t_n. Note that the column vector {P} is known at all times. In order to start a time-stepping scheme, the value a t = 0 (presumed to be given) is first used to calculate the solution at $t = t_1 = \Delta t$. This procedure is then repeated to obtain successive values of {u} at each time step.

Time stepping schemes such as these have inherent problems with accuracy and stability. Generally one can expect more accurate results if smaller time steps are used. However, in actual applications it is desired to take as large a time step as possible to minimize the computational expense. In addition to accuracy, larger time steps also can introduce some unwanted numerical oscillations into the solution. This is related to the stability of the numerical routine. Stability is concerned with the behavior of the solution as $t \to \infty$ while keeping the time step constant. In some case the numerical scheme is *conditionally stable* , i.e. is stable only for certain values of time step. Examples of typical behaviors are shown below



Unstable Stable (Oscillatory Decay) Stable (Monotonic Decay)

Consequently, an estimate for the largest time step for accurate and stable solutions is highly desirable. For all numerical routines in which $\alpha < ½$, the time stepping approximations are stable only if the time step satisfies the following stability relation

$$\Delta t < \Delta t_{cr} = \frac{2}{(1-2\alpha)\lambda} \tag{8}$$

where λ is the largest eigenvalue of the finite element equation (1). This is found by using a solution of the form $u(x,t) = U(x)e^{-\lambda t}$ in (1) to get $([B] - \lambda[A])\{U\} = \{P\}$

Approximations for Second-Order Time Derivatives

For second order time derivative problems, the spatial discretization will produce the system

$$[A]\{\ddot{u}\} + [B]\{u\} = \{F\} \tag{9}$$

There exists several approximation schemes available for this type of equation, and Bathe outlines several these techniques. These schemes are normally described as *direct integration methods*, in that the equations are integrated using a numerical stepping procedure without any transformation of the equations themselves. The direct integration procedure in based on two ideas. First, instead of attempting to satisfy the equations at any time t, the method satisfies the equation only at discrete time intervals. Secondly, the variation of the primary unknown along with its first and second time derivatives are assumed to be known over each time interval. Some of the more common techniques within this general scheme are

1. Central Difference Method
2. Houbolt Method
3. Wilson θ Method
4. Newmark Method

The most commonly used scheme is the *Newmark direct integration method.* In the Newmark direct integration method the first time derivative $\{\dot{u}\}$ and the function $\{u\}$ itself are approximated at the (n+1)th step $\Delta t_1 = \Delta t_2 = \ldots = \Delta t$) by the following

$$\begin{aligned} \{\dot{u}\}_{n+1} &= \{\dot{u}\}_n + [(1-\alpha)\{\ddot{u}\}_n + \alpha\{\ddot{u}\}_{n+1}]\Delta t \\ \{u\}_{n+1} &= \{u\}_n + \{\dot{u}\}_n \Delta t + [(\frac{1}{2} - \beta)\{\ddot{u}\}_n + \beta\{\ddot{u}\}_{n+1}](\Delta t)^2 \end{aligned} \tag{10}$$

where α and β are parameters that control the accuracy and the stability of the scheme, and the subscript *n* indicates that the solution is evaluated at the *n*th time step (i.e., at time $t = t_n$). For linear problems, the choice $\alpha = 1/2$ and $\beta = 1/4$ is known to given an unconditionally stable scheme, which corresponds to a *constant-average-acceleration method.* The case $\alpha = 1/2$ and $\beta = 1/6$ corresponds to a *linear acceleration method.* Other particular schemes are given in the text (see p 325).

Combining equations (9) and (10) yields

$$[\hat{A}]\{u\}_{n+1} = \{\hat{F}\}_{n,\,n+1} \tag{11}$$

where

$$\begin{aligned} [\hat{A}] &= [B] + a_0[A] \\ \{\hat{F}\} &= \{F\}_{n+1} + [A](a_0\{u\}_n + a_1\{\dot{u}\}_n + a_2\{\ddot{u}\}_n) \end{aligned} \tag{12}$$

Once the solution is known at t_{n+1}, the first and second derivatives of {u} at t_{n+1} can be computed from equation (10).

$$\{\ddot{u}\}_{n+1} = a_0(\{u\}_{n+1} - \{u\}_n) - a_1\{\dot{u}\}_n - a_2\{\ddot{u}\}_n$$

$$\{\dot{u}\}_{n+1} = \{\dot{u}\}_n + a_3\{\ddot{u}\}_n + a_4\{\ddot{u}\}_{n+1} \tag{13}$$

where

$$a_0 = \frac{1}{\beta\Delta t^2}, \quad a_1 = a_0\Delta t, \quad a_2 = \frac{1}{2\beta} - 1, \quad a_3 = (1-\alpha)\Delta t, \quad a_4 = \alpha$$

For a given set of initial conditions $\{u\}_0$, $\{\dot{u}\}_0$, and $\{\ddot{u}\}_0$, we can solve (13) repeatedly, marching forward in time, for the solution {u} and its time derivatives at any time $t > 0$.

The various schemes within the Newmark method are generally *stable*. For all schemes in which $2\beta < \alpha$, the stability requirement is given by the relation

$$\Delta t \le \Delta t_{cr} = [\tfrac{1}{2}\omega_{max}^2(\alpha - 2\beta)]^{-\frac{1}{2}} \tag{14}$$

where ω_{max} is the maximum natural frequency of the system $\left([B] - \omega^2[A]\right) = \{F\}$

Initial values of {ü} are generally not known from the problem description. For this case one can make use of equation (9) at the time $t = 0$ to compute {ü}.

One-Dimensional Time-Dependent Problems

Consider the general one-dimensional time dependent equation,

$$c_1\frac{\partial u}{\partial t} + c_2\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}) + \frac{\partial^2}{\partial x^2}(b\frac{\partial^2 u}{\partial x^2}) + f = 0, \quad 0 < x < L \tag{15}$$

Clearly both second and fourth order problems are contained in this general equation. Following the standard technique, we construct the weak form

$$\int_{x_e}^{x_{e+1}} v\,[c_1\frac{\partial u}{\partial t} + c_2\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(a\frac{\partial u}{\partial x}) + \frac{\partial^2}{\partial x^2})b\frac{\partial^2}{\partial x^2}) + f]dx =$$

$$\int_{x_e}^{x_{e+1}}(c_1 v\frac{\partial u}{\partial t} + c_2 v\frac{\partial^2 u}{\partial t^2} + a\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + b\frac{\partial^2}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + vf)dx =$$

$$+ \{v\,[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(b\frac{\partial^2 u}{\partial x^2})]\}_{x=x_e}^{x=x_{e+1}} - [b\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2}]_{x=x_e}^{x=x_{e+1}} = 0$$

which can be written as

$$\int_{x_e}^{x_{e+1}}(c_1 v\frac{\partial u}{\partial t} + c_2 v\frac{\partial^2 u}{\partial t^2} + a\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + b\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + vf)\,dx$$

$$- \hat{Q}_1 v(x_e) - \hat{Q}_3 v(x_{e+1}) - \hat{Q}_2[-\frac{dv}{dx}(x_e)] - \hat{Q}_4[-\frac{dv}{dx}(x_{e+1})] = 0 \tag{16}$$

where

$$\hat{Q}_1 = [-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(b\frac{\partial^2 u}{\partial x^2})]_{x=x_e}, \qquad \hat{Q}_2^{(e)} = [b\frac{\partial^2 u}{\partial x^2}]_{x=x_e}$$

$$\hat{Q}_3 = [a\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(b\frac{\partial^2 u}{\partial x^2})]_{x=x_{e+1}}, \qquad \hat{Q}_4^{(e)} = [b\frac{\partial^2 u}{\partial x^2}]_{x=x_{e+1}} \tag{17}$$

Assuming that the time and spatial variations of u are separable, we follow the Ritz scheme and let

$$u = \sum_{j=1}^{r} u_j(t)\psi_j(x), \qquad v = \psi_i(x) \tag{18}$$

Equations (18) with (16) then give

$$\int_{x_e}^{x_{e+1}} (c_1\psi_i \sum_{j=1}^{r} \frac{du_j}{dt}\psi_j + c_2\psi_i \sum_{j=1}^{r} \frac{d^2u_j}{dt^2}\psi_j + a\frac{d\psi_i}{dx}\sum_{j=1} u_j\frac{d\psi_j}{dx} + b\frac{d^2\psi_i}{dx^2}\sum_{j=1}^{r} u_j\frac{d^2\psi_j}{dx^2} + \psi_i f)\, dx - \hat{Q}_i = 0 \tag{19}$$

or

$$[M^1]\{\dot{u}\} + [M^2]\{\ddot{u}\} + ([K^1] + [K^2])\{u\} = \{F\} \tag{20}$$

where

$$M_{ij}^1 = \int_{x_e}^{x_{e+1}} c_1\psi_i\psi_j\, dx, \quad M_{ij}^2 = \int_{x_e}^{x_{e+1}} c_2\psi_i\psi_j\, dx$$

$$K_{ij}^1 = \int_{x_e}^{x_{e+1}} a\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\, dx, \quad K_{ij}^2 = \int_{x_e}^{x_{e+1}} b\frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}\, dx \tag{21}$$

$$F_i = -\int_{x_e}^{x_{e+1}} \psi_i f\, dx + \hat{Q}_i$$

Consider the special case of the first order time derivative example

$$[M^1]\{\dot{u}\} + [K]\{u\} = \{F\} \tag{22}$$

Using the previous time stepping scheme yields the matrix equation

$$[\hat{K}^1]\{u\}_{n+1} = \{\hat{F}^1\} \tag{23}$$

where

$$
\begin{aligned}
[\hat{K}^1] &= [M^1]+\alpha\Delta t[K] \\
\{\hat{F}^1\} &= ([M^1]-(1-\alpha)\Delta t\ [K])\{u\}_n \\
&\quad +\Delta t\ (\alpha\{F\}_{n+1}+(1-\alpha)\{F\}_n)
\end{aligned} \tag{24}
$$

Likewise the special case of a second order time derivative example would be

$$
[M^2]\{\ddot{u}\} + [K]\{u\} = \{F\} \tag{25}
$$

Using the Newmark time stepping scheme produces the system

$$
[\hat{K}^2]\{u\}_{n+1} = \{\hat{F}^2\} \tag{26}
$$

where

$$
\begin{aligned}
[\hat{K}^2] &= [K] + a_o[M^2] \\
\{\hat{F}^2\} &= \{F\}_{n+1}+ [M^2](a_0\{u\}_n+ a_1\{\dot{u}\}_n+ a_2\{\ddot{u}\}_n) \\
\{\ddot{u}\}_0 &= [M^2]^{-1}(\{F\} - [K]\{u\}_0)
\end{aligned} \tag{27}
$$

Mass Lumping

For time dependent problems considered here, the matrices that multiply either the first or second order time derivatives of the primary nodal unknowns are generally called *mass matrices*, see for example relations (22) or (25). These matrices were developed from the standard variational/weak form procedure, and are commonly called *consistent mass matrices.* In general they are symmetric, positive definite and *non-diagonal.* Since solution procedures normally require inverting these matrices, it is computationally advantageous to have these matrices be diagonal. Several methods have been developed to construct such diagonal mass matrices (see text pp 326-328). One method is to use a *row-sum lumping* scheme whereby the sum of elements in each row in the consistent matrix are used as the diagonal elements. Considering the one-dimensional consistent mass matrix $[M] = \int_{x_a}^{x_b} \rho\psi_i\psi_j dx$, the linear and quadratic consistent and lumped mass matrices are given by

$$
\begin{aligned}
[M] &= \frac{\rho h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow [M]_L = \frac{\rho h_e}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
[M] &= \frac{\rho h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \Rightarrow [M]_L = \frac{\rho h_e}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{28}
$$

Use of mass lumping will also effect the critical time step required for conditional stability and in fact will generally lead to a larger critical time step (see text pp 327-328).