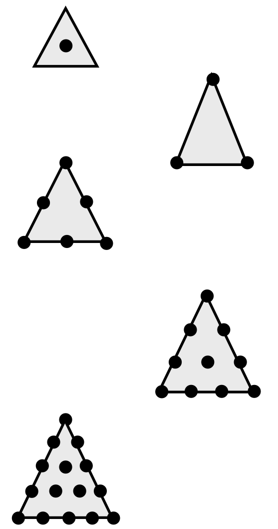
CHAPTER 9

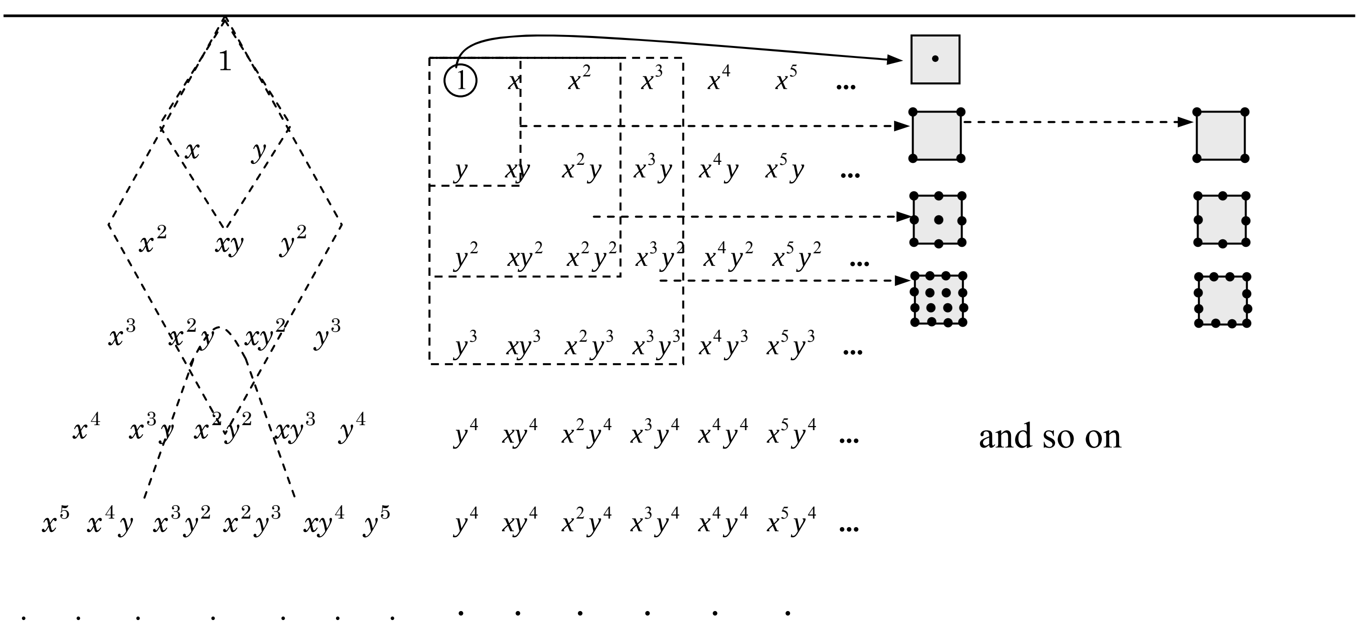
- **Interpolation functions for 2D elements**
- **Numerical Integration**
- **Modeling Considerations**

Triangular Elements

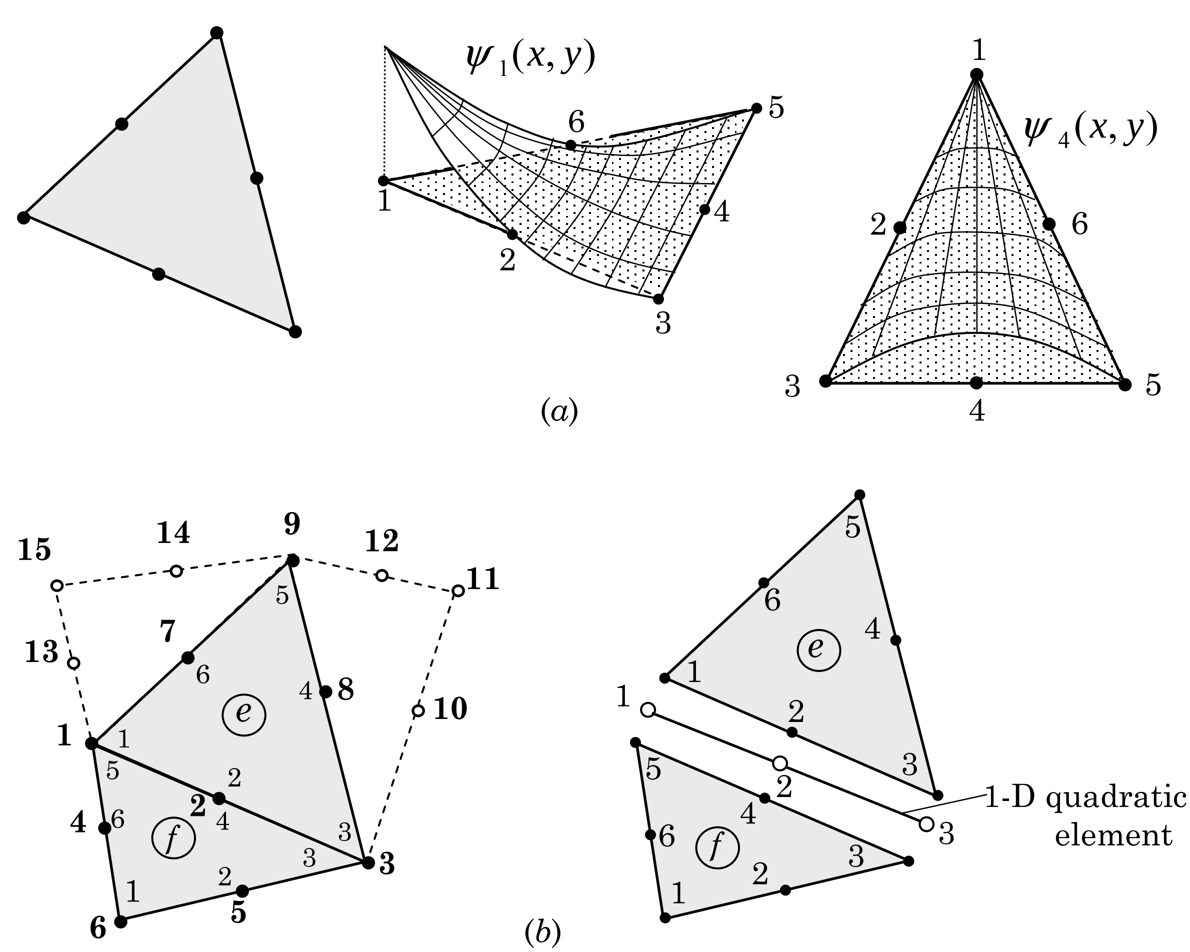
Pascal's triangle	Degree of the complete polynomial	Number of terms in the polynomial	Element with nodes
1	0	1	
$x \quad y$	1	3	
$x^2 \quad xy \quad y^2$	2	6	
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3	10	
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4	15	
$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$	5	21	(Figure not shown)



Lagrange and Serendipity Elements

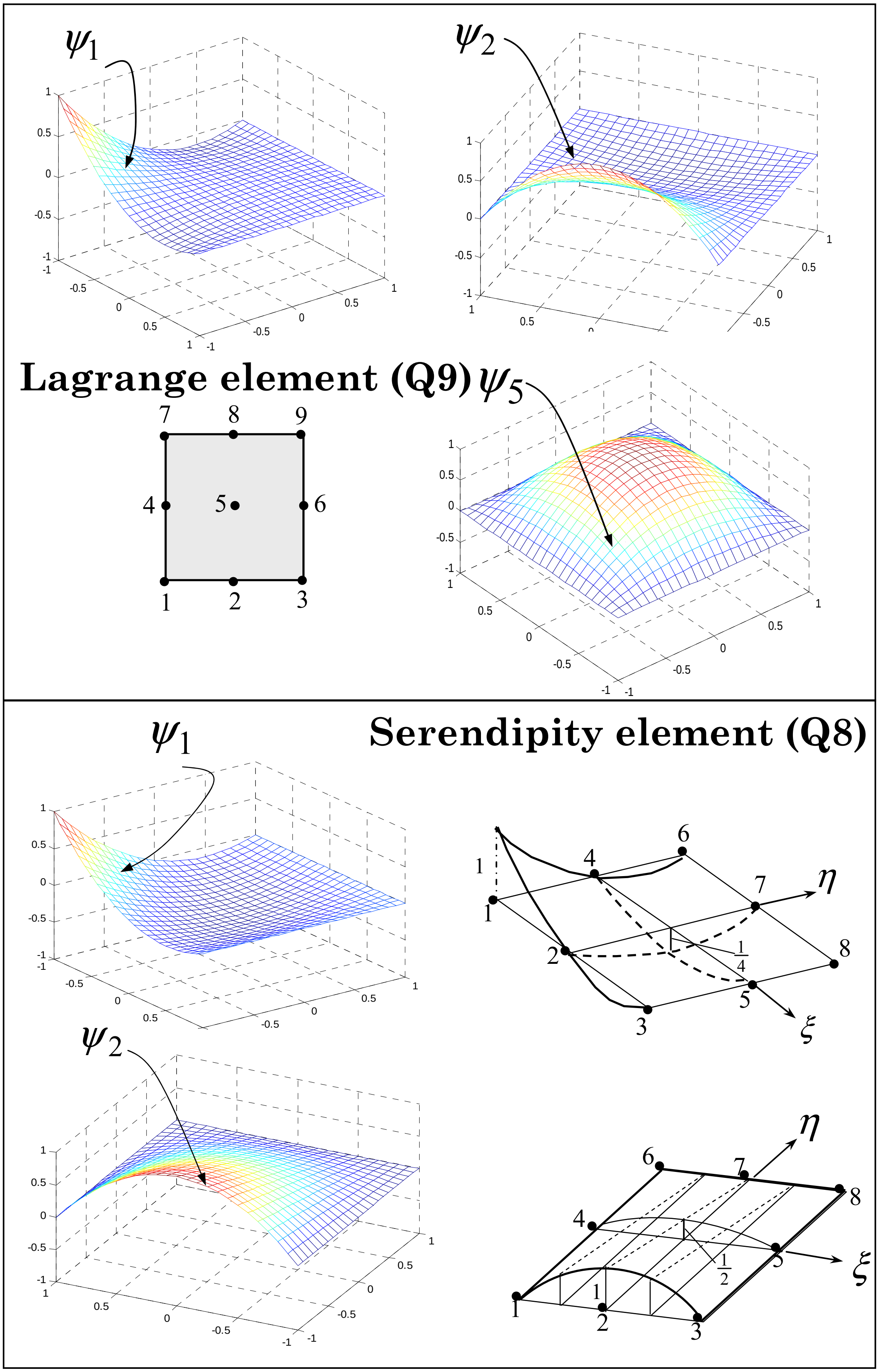


Lagrange element (T6)



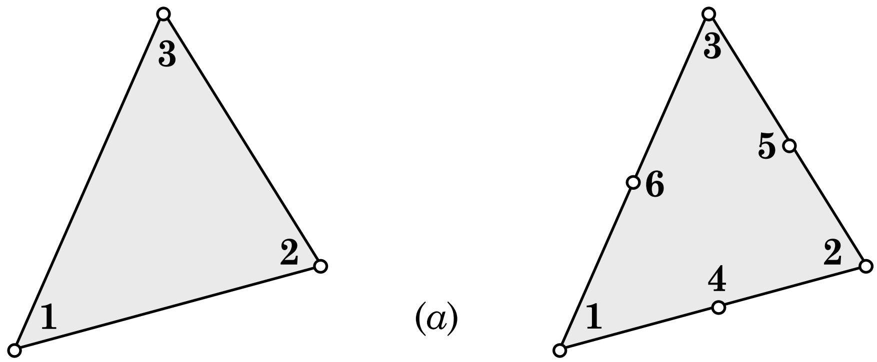
(a)

(b)


ψ_1
ψ_2
Lagrange element (Q9)
ψ_5
7 8 9
4 5 6
1 2 3
ψ_1
Serendipity element (Q8)
ψ_2
1
$\frac{1}{4}$
η
ξ
$\frac{1}{2}$

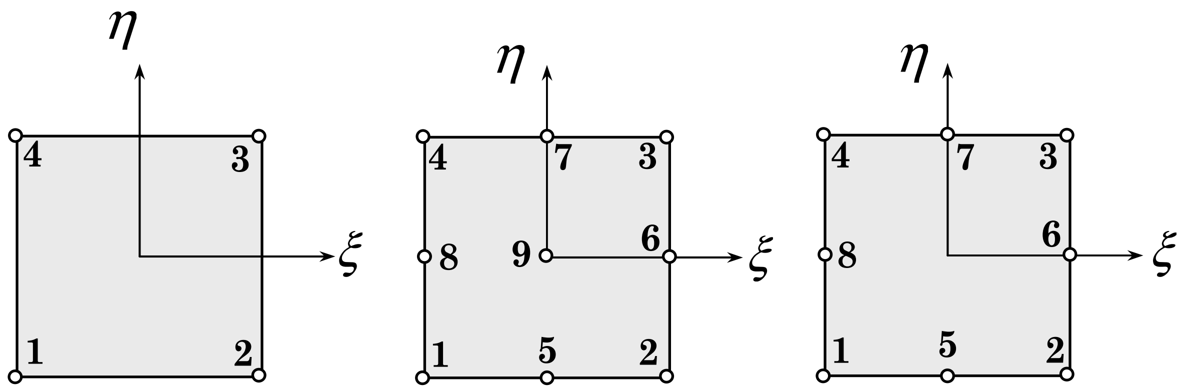
Some Triangular and Rectangular Elements

Lagrange triangular elements



(a)

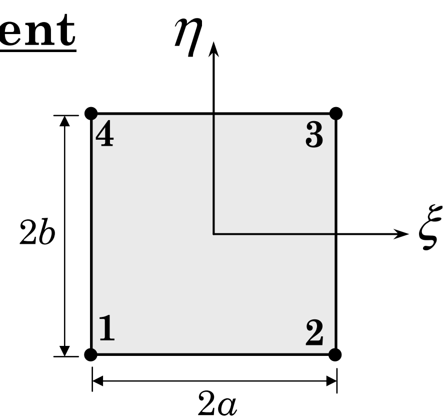
Lagrange rectangular elements



(b)

Hermite cubic rectangular element

- ∘ Nodes with function values only (u)
- • Nodes with values of the function (u) and its derivatives $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y}\right)$



(c)

Numerical Evaluation of Element Coefficients

Governing Equation

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\left[a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}\right]\\
&-\frac{\partial}{\partial y}\left[b(x,y)\frac{\partial u}{\partial x}+c(x,y)\frac{\partial u}{\partial y}\right]+d(x,y)u=f(x,y)
\end{aligned}
$$

Finite Element Approximation

$$
u(x,y)=\sum_{j=1}^{n}u_j^e\psi_j^e(x,y)
$$

Finite Element Model

$$
[K^e]\{u^e\}=\{f^e\}+\{Q^e\}
$$

$$
\begin{aligned}
K_{ij}^e=\int_{\Omega^e}\Bigg[&a(x,y)\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial x}+b(x,y)\left(\frac{\partial\psi_i^e}{\partial x}\frac{\partial\psi_j^e}{\partial y}+\frac{\partial\psi_j^e}{\partial x}\frac{\partial\psi_i^e}{\partial y}\right)\\
&+c(x,y)\frac{\partial\psi_i^e}{\partial y}\frac{\partial\psi_j^e}{\partial y}+d(x,y)\psi_i^e\psi_j^e\Bigg]dxdy
\end{aligned}
$$

$$
f_i^e=\int_{\Omega^e}f(x,y)\psi_i^e\ dxdy,\quad Q_i^e=\oint_{\Gamma^e}q_n(s)\psi_i^e\ ds
$$

$$
q_n=\left(a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}\right)n_x+\left(b\frac{\partial u}{\partial x}+c\frac{\partial u}{\partial y}\right)n_y
$$

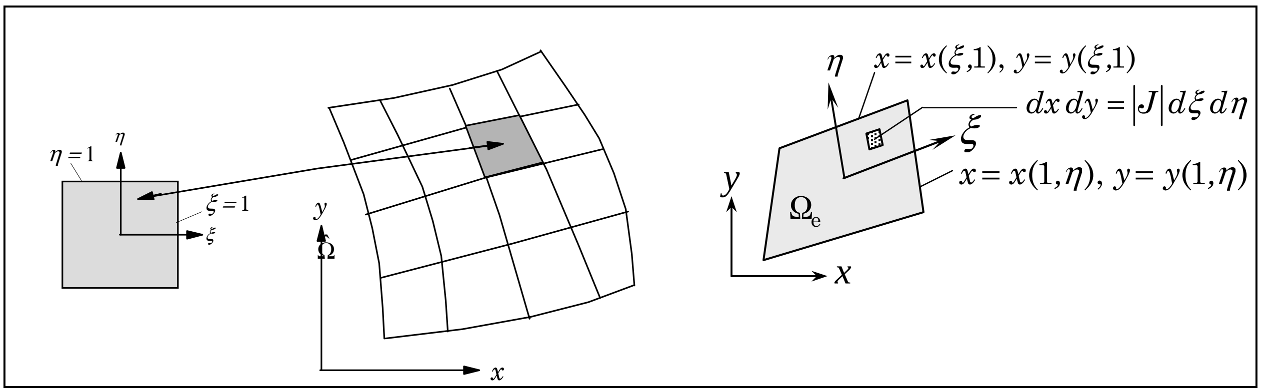
Subparametric, Isoparametric, and Superparametric Formulations

Geometry:

$$x = \sum_{j=1}^{m} x_j^e \hat{\psi}_j^e(\xi, \eta) \; , \qquad y = \sum_{j=1}^{m} y_j^e \hat{\psi}_j^e(\xi, \eta)$$

Solution:

$$u(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x, y) = \sum_{j=1}^{n} u_j^e \psi_j^e(x(\xi, \eta), y(\xi, \eta))$$



1. *Superparametric* ($m > n$): The polynomial degree of approximation used for the geometry is of higher order than that used for the dependent variable.

2. *Isoparametric* ($m = n$): Equal degree of approximation is used for both geometry and dependent variables.

3. *Subparametric* ($m < n$): Higher-order approximation of the dependent variable is used.

<u>Gauss Quadrature</u>

$$\int_{\Omega^e} F(x,y)\ dxdy = \int_{\hat{\Omega}} F(\xi,\eta) J(\xi,\eta)\ d\xi d\eta$$

$$= \int_{\hat{\Omega}} \hat{F}(\xi,\eta)\ d\xi d\eta$$

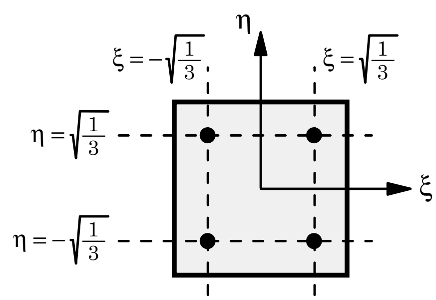
$$= \sum_{I=1}^{N_\xi} \sum_{J=1}^{N_\eta} \hat{F}(\xi_I,\eta_J) W_I W_J$$

Domain of the *physical* element

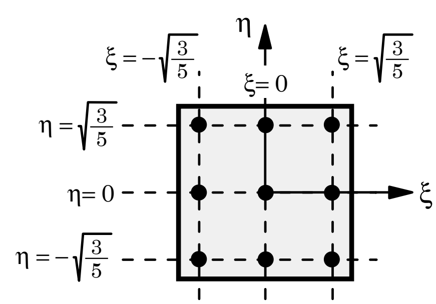
Domain of the *master* element

Gauss *points*

Gauss *weights*

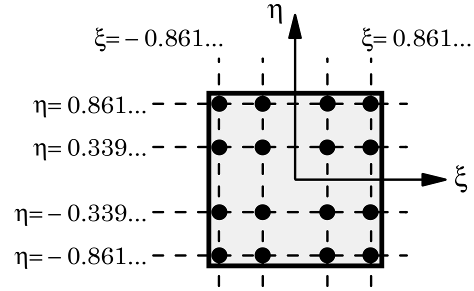


$$\left\{ \begin{array}{c} \dfrac{\partial \psi_i^e}{\partial \xi} \\ \dfrac{\partial \psi_i^e}{\partial \eta} \end{array} \right\} = \left[\begin{array}{cc} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{array} \right]^e \left\{ \begin{array}{c} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial y} \end{array} \right\}$$



$$[J]^e = \left[\begin{array}{cc} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{array} \right]^e$$

$$\left\{ \begin{array}{c} \dfrac{\partial \psi_i^e}{\partial x} \\ \dfrac{\partial \psi_i^e}{\partial y} \end{array} \right\} = [J]^{-1} \left\{ \begin{array}{c} \dfrac{\partial \psi_i^e}{\partial \xi} \\ \dfrac{\partial \psi_i^e}{\partial \eta} \end{array} \right\}$$



Element Calculations Using Numerical Quadrature

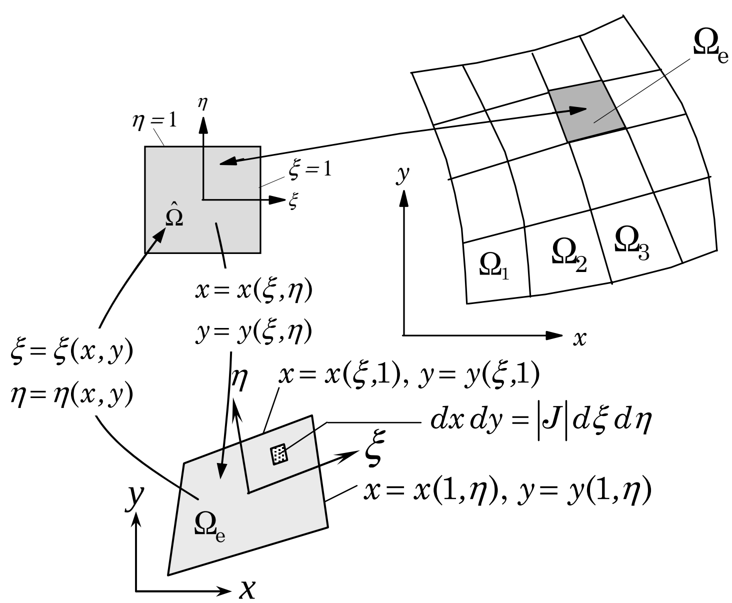
$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \xi} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \xi} \\ \sum_{i=1}^{m} x_i \frac{\partial \hat{\psi}_i}{\partial \eta} & \sum_{i=1}^{m} y_i \frac{\partial \hat{\psi}_i}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial \hat{\psi}_1}{\partial \xi} & \frac{\partial \hat{\psi}_2}{\partial \xi} & \cdots & \frac{\partial \hat{\psi}_m}{\partial \xi} \\ \frac{\partial \hat{\psi}_1}{\partial \eta} & \frac{\partial \hat{\psi}_2}{\partial \eta} & \cdots & \frac{\partial \hat{\psi}_M}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_m & y_m \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial \psi_i^e}{\partial x} \\ \frac{\partial \psi_i^e}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial \psi_i^e}{\partial \xi} \\ \frac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix} \equiv [J^*] \begin{Bmatrix} \frac{\partial \psi_i^e}{\partial \xi} \\ \frac{\partial \psi_i^e}{\partial \eta} \end{Bmatrix}$$

$$J_{11}^* = \frac{J_{22}}{J} \ , \quad J_{12}^* = -\frac{J_{12}}{J} \ , \quad J_{22}^* = \frac{J_{11}}{J} \ , \quad J_{21}^* = -\frac{J_{21}}{J}$$

$$J = J_{11}J_{22} - J_{12}J_{21}$$



Evaluation of Element Coefficient Matrix Using the Gauss Quadrature (continued)

$$
\begin{aligned}
K_{ij}^e = \int_{\hat{\Omega}} \Bigg\{ & a(\xi,\eta)\left(J_{11}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_i^e}{\partial \eta}\right)\left(J_{11}^* \frac{\partial \psi_j^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_j^e}{\partial \eta}\right) \\
& + b(\xi,\eta)\left[\left(J_{11}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_i^e}{\partial \eta}\right)\left(J_{21}^* \frac{\partial \psi_j}{\partial \xi} + J_{22}^* \frac{\partial \psi_j}{\partial \eta}\right)\right. \\
& + \left.\left(J_{11}^* \frac{\partial \psi_i^e}{\partial \xi} + J_{12}^* \frac{\partial \psi_i^e}{\partial \eta}\right)\left(J_{21}^* \frac{\partial \psi_j}{\partial \xi} + J_{22}^* \frac{\partial \psi_j}{\partial \eta}\right)\right] \\
& + c(\xi,\eta)\left(J_{21}^* \frac{\partial \psi_j}{\partial \xi} + J_{22}^* \frac{\partial \psi_j}{\partial \eta}\right)\left(J_{21}^* \frac{\partial \psi_j}{\partial \xi} + J_{22}^* \frac{\partial \psi_j}{\partial \eta}\right) \\
& + d(\xi,\eta)\psi_i^e \psi_j^e \Bigg\} J \; d\xi d\eta \equiv \int_{\hat{\Omega}} \hat{F}_{ij}(\xi,\eta) d\xi d\eta
\end{aligned}
$$

$$
\begin{aligned}
\int_{\hat{\Omega}} \hat{F}_{ij}(\xi,\eta) d\xi d\eta &= \int_{-1}^{1}\int_{-1}^{1} \hat{F}_{ij}(\xi,\eta) d\eta d\xi \\
&\approx \sum_{I=1}^{N_\xi}\sum_{J=1}^{N_\eta} \hat{F}_{ij}(\xi_I,\eta_J) W_I W_J
\end{aligned}
$$

$$
N_\xi = \text{int}[\frac{1}{2}(p+1)], \quad N_\eta = \text{int}[\frac{1}{2}(q+1)]
$$

Fortran Statements to Compute Element Coefficient Matrix and Source Vector

```
      SUBROUTINE ELEMKF(NPE,NN,INTF)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/STF/ELF(27),ELK(27,27),ELXY(9,2)
      COMMON/PST/A0,AX,AY,B0,BX,BY,C0,CX,CY,F0,FX,FY
      COMMON/SHP/SF(9),GDSF(2,9)
      DIMENSION  GAUSPT(5,5),GAUSWT(5,5)
C
      DATA GAUSPT/5*0.0D0, -0.57735027D0, 0.57735027D0, 3*0.0D0,
     2  -0.77459667D0, 0.0D0, 0.77459667D0, 2*0.0D0, -0.86113631D0,
     3  -0.33998104D0, 0.33998104D0, 0.86113631D0, 0.0D0, -0.90617984D0,
     4  -0.53846931D0,0.0D0,0.53846931D0,0.90617984D0/
C
      DATA GAUSWT/2.0D0, 4*0.0D0, 2*1.0D0, 3*0.0D0, 0.55555555D0,
     2   0.88888888D0, 0.55555555D0, 2*0.0D0, 0.34785485D0,
     3 2*0.65214515D0, 0.34785485D0, 0.0D0, 0.23692688D0,
     4   0.47862867D0, 0.56888888D0, 0.47862867D0, 0.23692688D0/
C
      NDF = NN/NPE
C
C     Initialize the arrays
C
      DO 120 I = 1,NN
      ELF(I)  = 0.0
      DO 120 J = 1,NN
  120 ELK(I,J)= 0.0
C
C     Do-loops on numerical (Gauss) integration begin here.
C     Subroutine SHPRCT (SHaPe functions for ReCTangular
C     elements) is called here
C
      DO 200 NI = 1,INTF
      DO 200 NJ = 1,INTF
      XI  = GAUSPT(NI,INTF)
      ETA = GAUSPT(NJ,INTF)
      CALL SHPRCT (NPE,XI,ETA,DET,ELXY)
      CNST = DET*GAUSWT(NI,INTF)*GAUSWT(NJ,INTF)
```

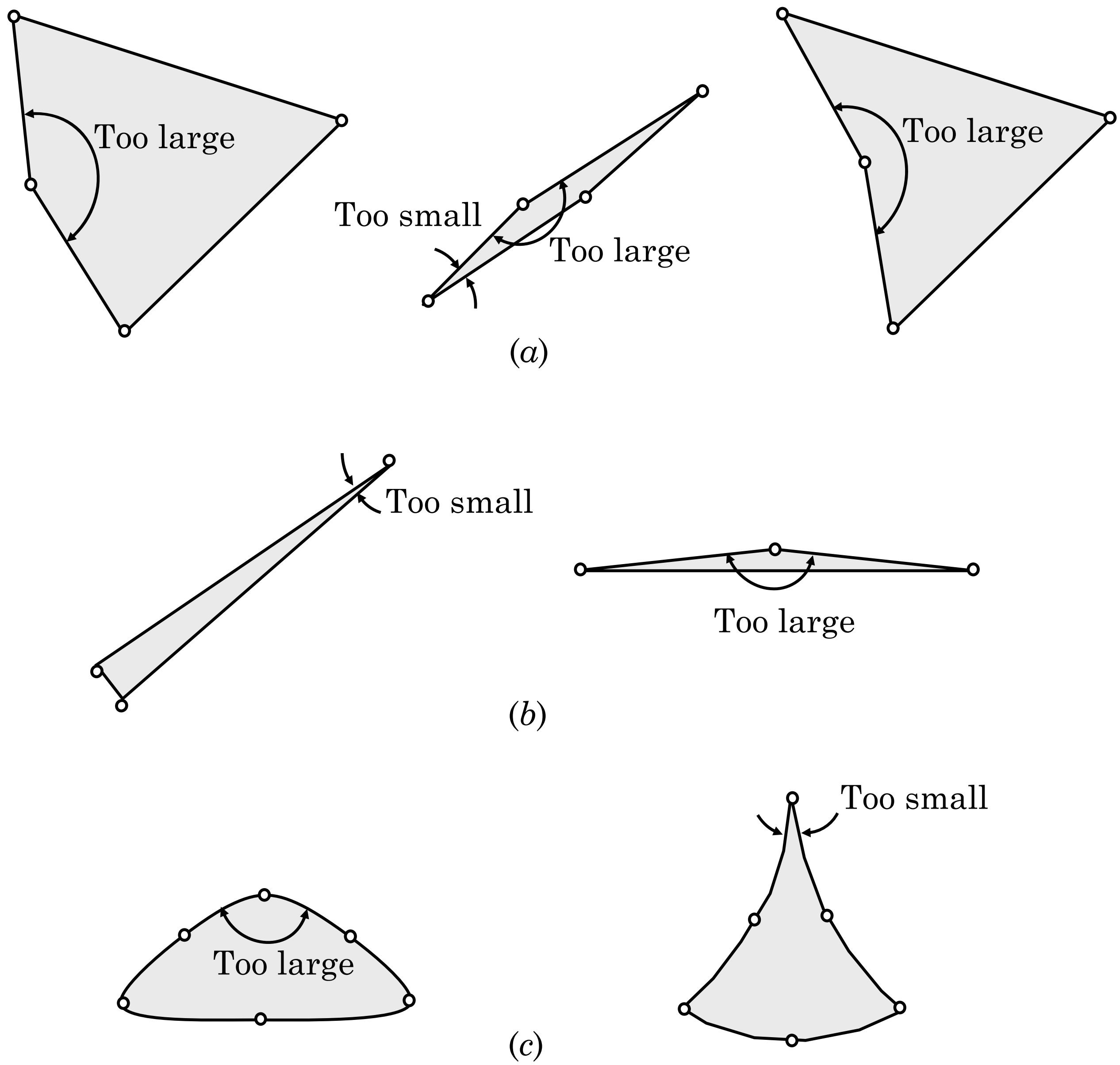
Fortran Statements to Compute Element Coefficient Matrix and Source Vector (continued)

```
      X=0.0
      Y=0.0
      DO 140 I=1,NPE
      X=X+ELXY(I,1)*SF(I)
  140 Y=Y+ELXY(I,2)*SF(I)
C
      SOURCE=F0+FX*X+FY*Y
                AA=A0+AX*X+AY*Y
                BB=B0+BX*X+BY*Y
                CC=C0+CX*X+CY*Y
C
      II=1
      DO 180 I=1,NPE
      JJ=1
      DO 160 J=1,NPE
        S00=SF(I)*SF(J)*CNST
        S11=GDSF(1,I)*GDSF(1,J)*CNST
        S22=GDSF(2,I)*GDSF(2,J)*CNST
        S12=GDSF(1,I)*GDSF(2,J)*CNST
        S21=GDSF(2,I)*GDSF(1,J)*CNST
C
        ELK(I,J) = ELK(I,J) + AA*S11 + BB*(S12 + S21) + CC*S22
C
  160 JJ = NDF*J+1
C
        ELF(I) = ELF(I)+CNST*SF(I)*SOURCE
C
  180 II = NDF*I+1
  200 CONTINUE
        RETURN
        END
```

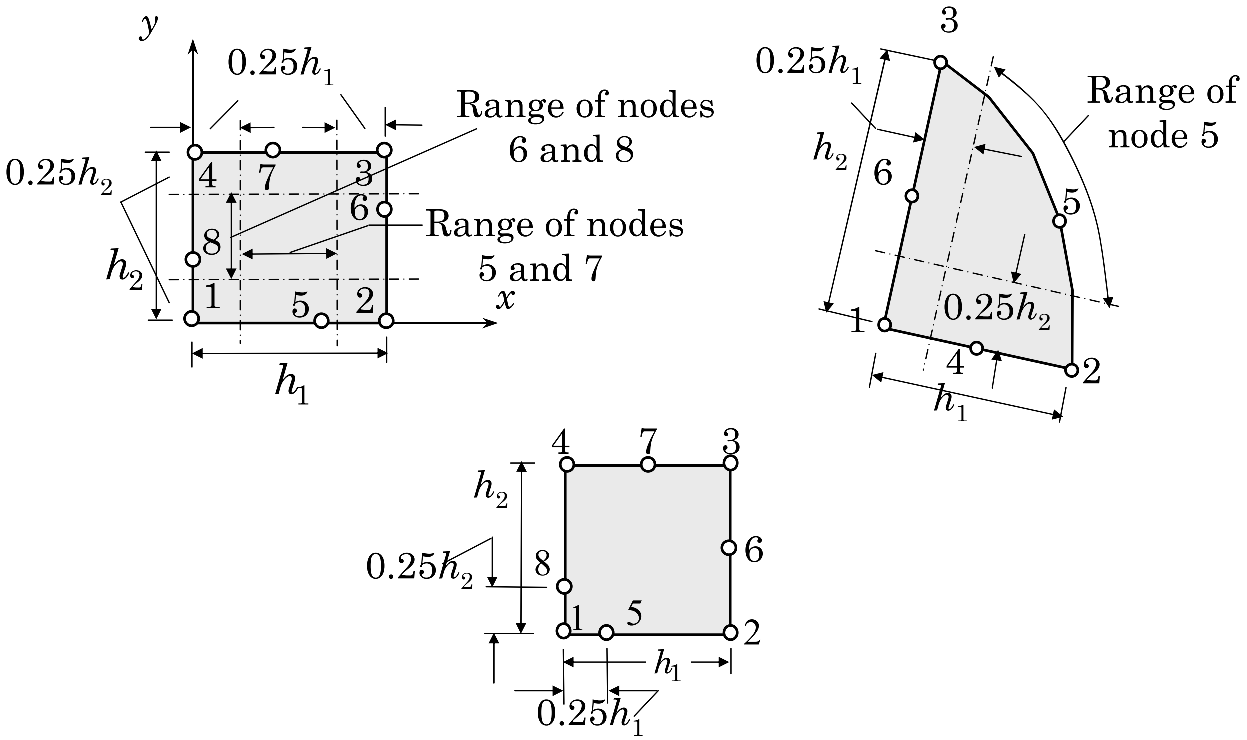
<u>MODELING CONSIDERATIONS</u>

- Elements with unacceptable vertex angles
- Mesh orientation and mesh refinements
- *h* and *p* refinements, and acceptable and unacceptable mesh refinements
- Range of acceptable location of the `midside' nodes
- Acceptable (compatible) connections between elements of different kind
- Various types of incompatible connections
- Mesh refinements with compatible and incompatible connections
- Handling of mathematical singularities in the application of boundary conditions

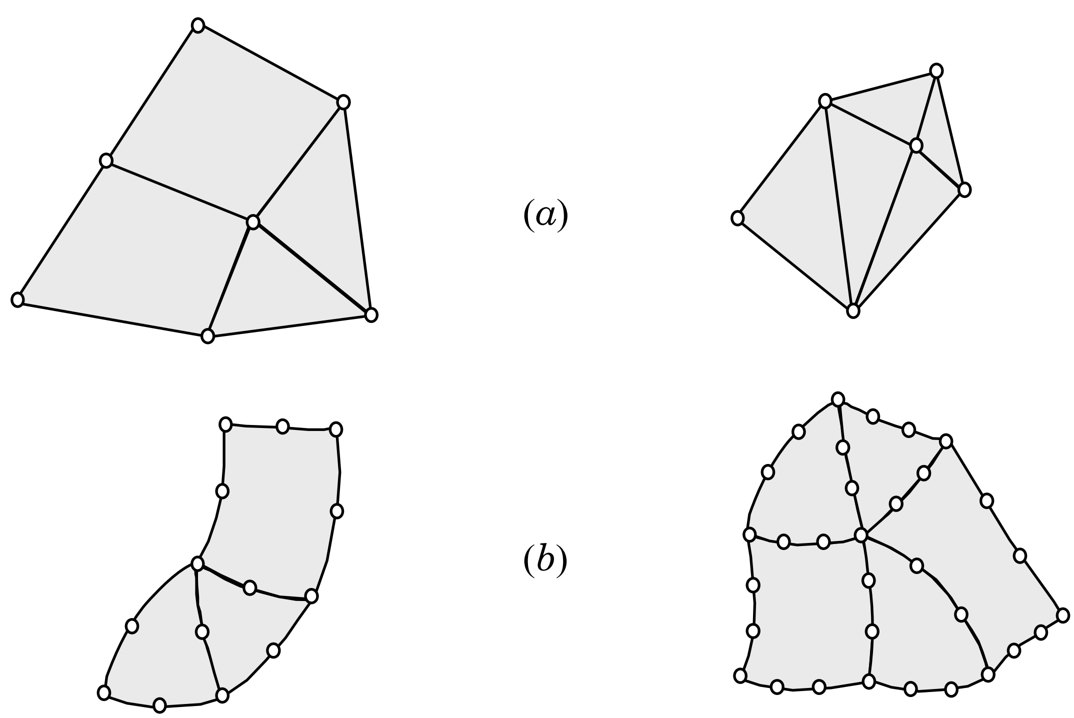
Unacceptable Vertex Angles



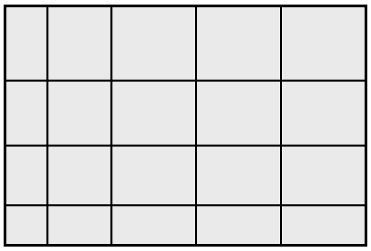
Placement of Midside Nodes



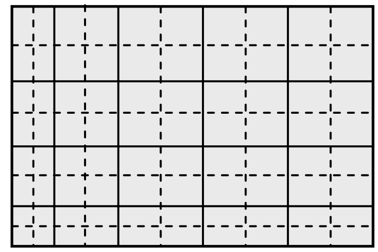
Compatible Connections Between Elements



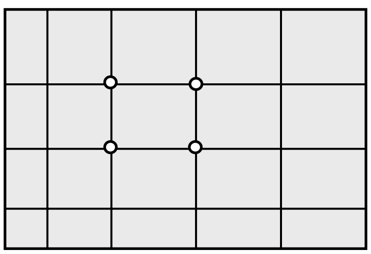
Mesh Refinements



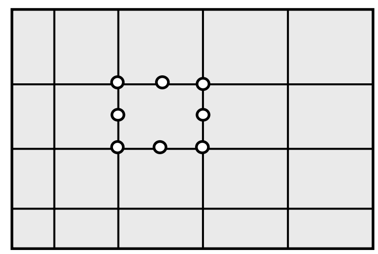
(*a*)



A *mesh refinement* must contain the previous mesh as a subset



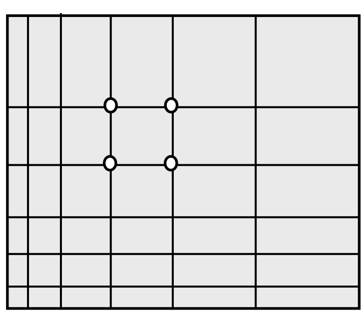
(*b*)



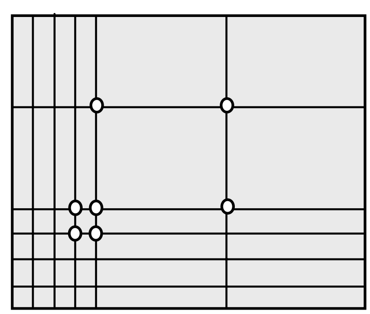
h *refinement*

p *refinement*

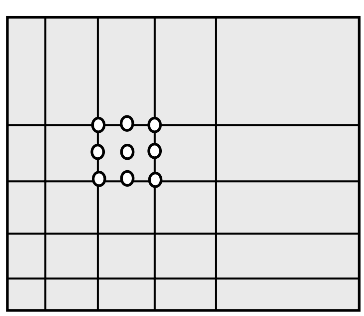
(*a*)



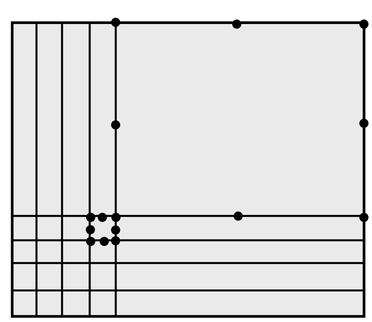
(*b*)

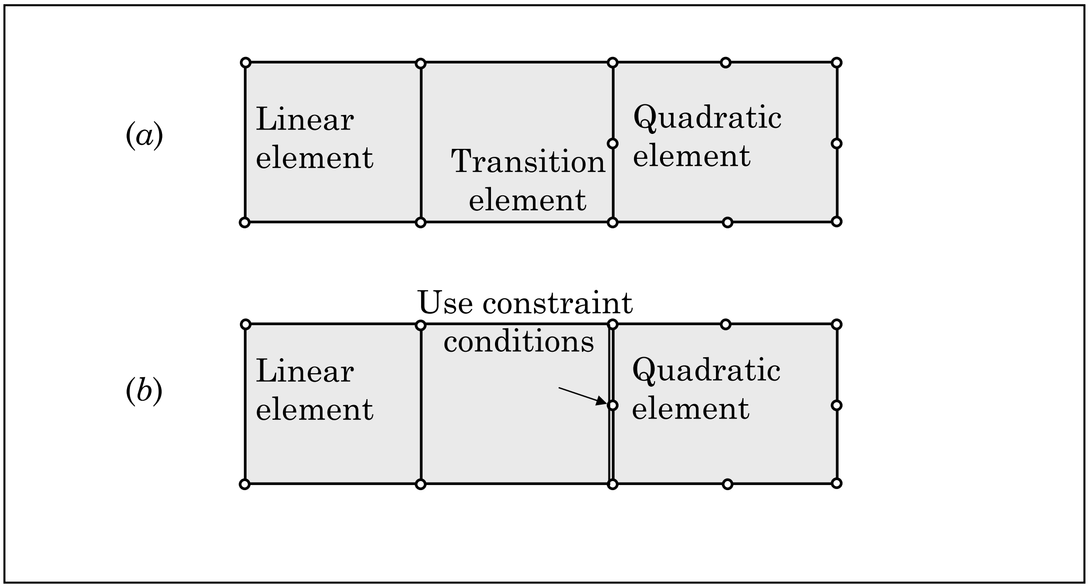


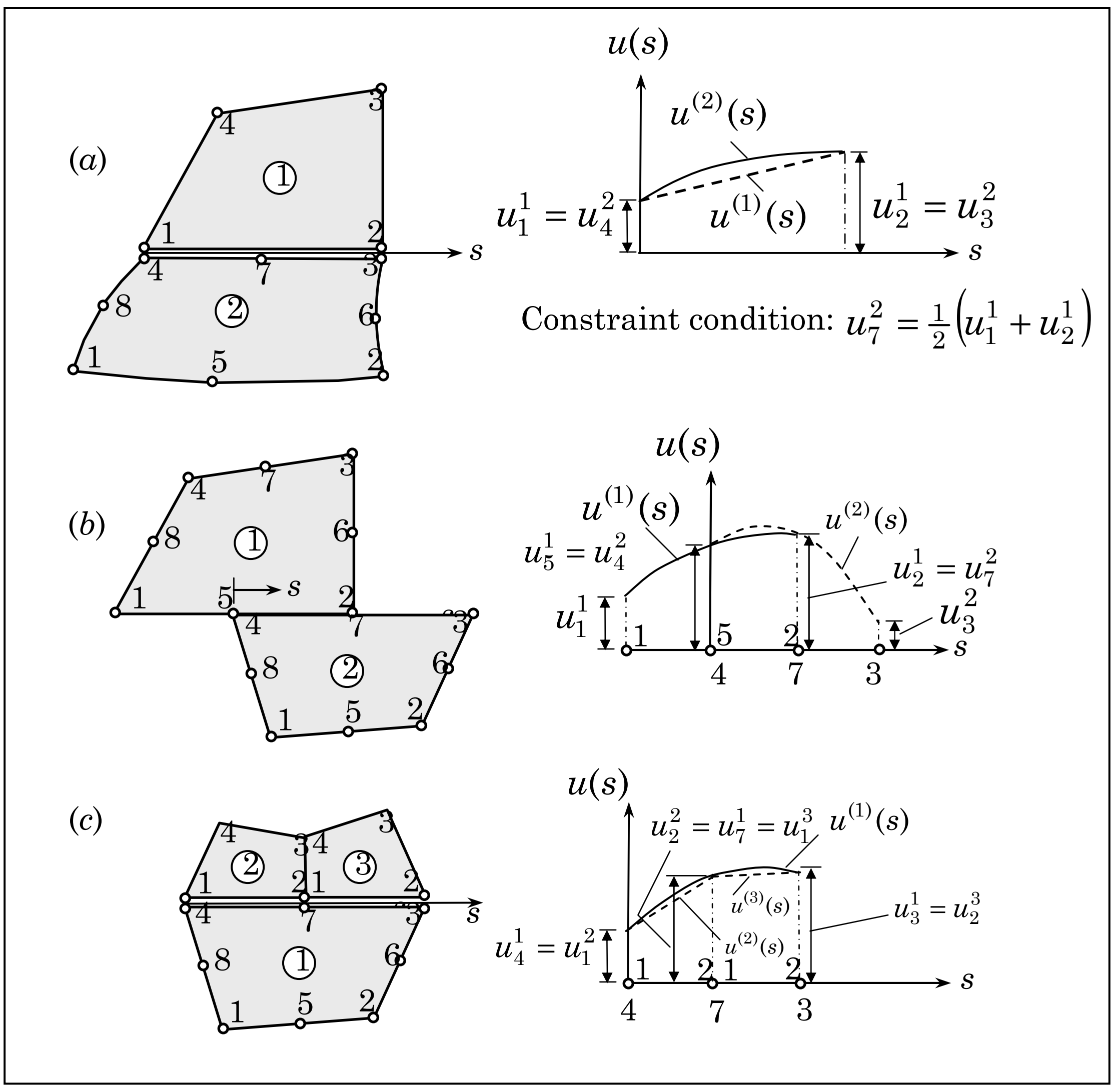
(*c*)



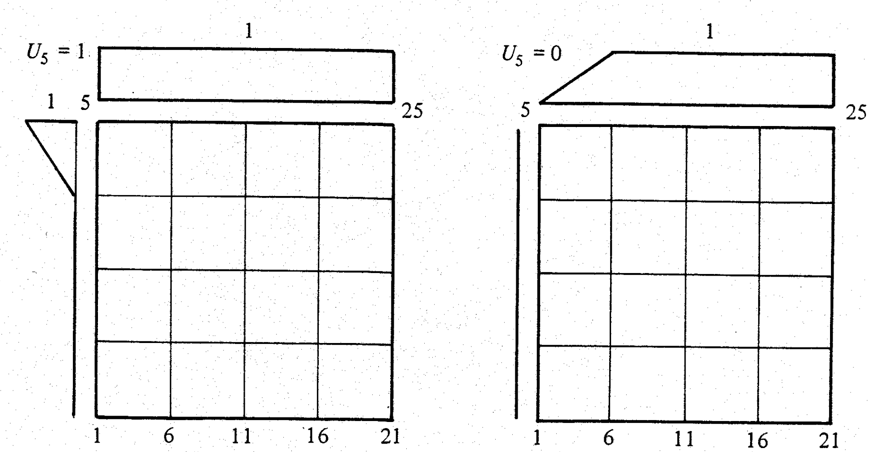
(*d*)



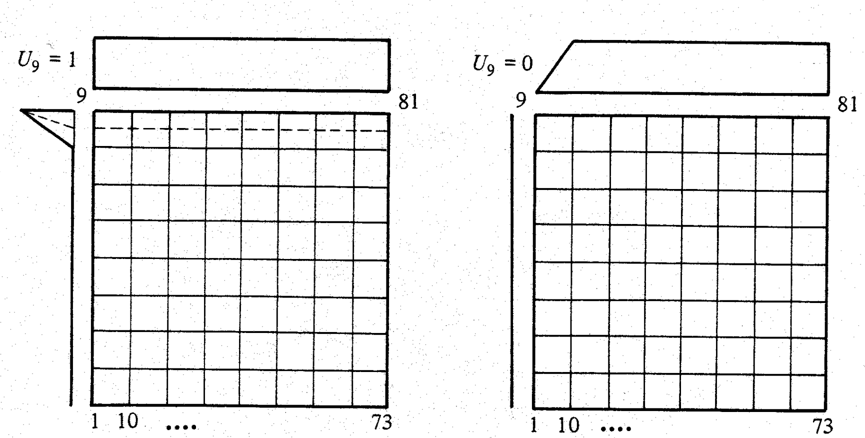

(a)
Linear element
Transition element
Quadratic element
(b)
Use constraint conditions
Linear element
Quadratic element


(a)
u(s)
u^(2)(s)
u^(1)(s)
u_1^1 = u_4^2
u_2^1 = u_3^2
s
Constraint condition: u_7^2 = 1/2 (u_1^1 + u_2^1)
(b)
u(s)
u^(1)(s)
u^(2)(s)
u_5^1 = u_4^2
u_2^1 = u_7^2
u_1^1
u_3^2
s
(c)
u(s)
u_2^2 = u_7^1 = u_1^3
u^(1)(s)
u^(3)(s)
u^(2)(s)
u_3^1 = u_2^3
u_4^1 = u_1^2
s

Handling of Mathematical Singularities in Applying Boundary Conditions



(*a*) 4 X 4 mesh (mesh 1) (*b*)



(*c*) 8 X 8 mesh (mesh 2) (*d*)