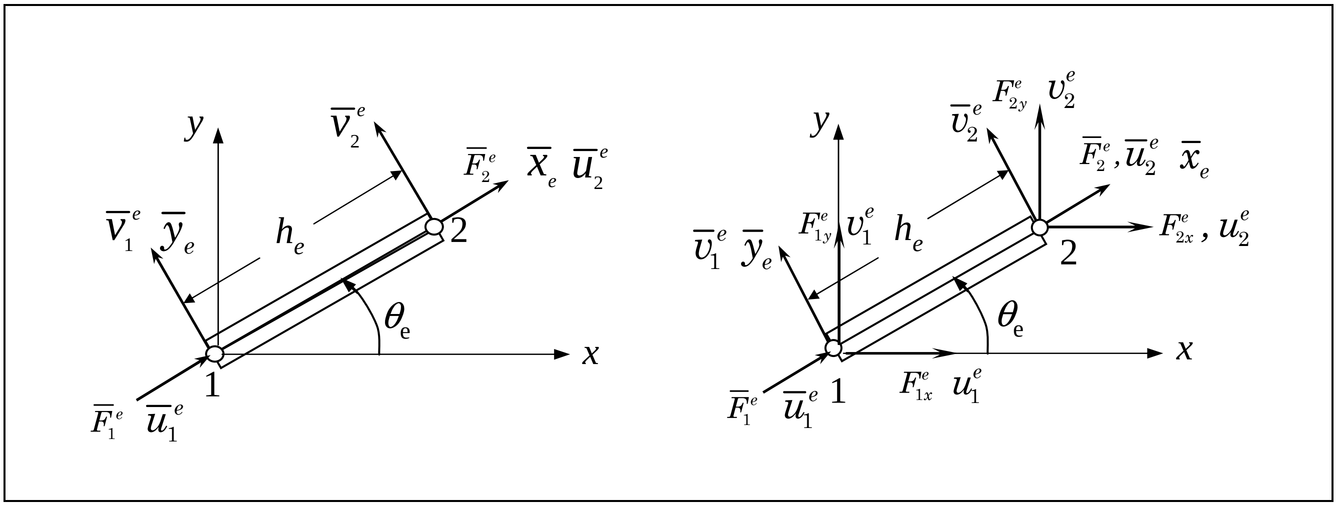
Plane Truss Finite Elements



Truss Finite Element Equations

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1^e \\ 0 \\ \bar{F}_2^e \\ 0 \end{Bmatrix}$$

$$[\bar{K}^e]\{\bar{\Delta}^e\} = \{\bar{F}^e\}$$

Transformation Equations

$$\bar{x}_e = x\,\cos\theta + y\,\sin\theta$$

$$\bar{y}_e = -x\,\sin\theta + y\,\cos\theta$$

$$\begin{Bmatrix} \bar{x}_e \\ \bar{y}_e \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix}\begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix}$$

$$\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\}$$

Transformation Element Stiffness Matrix and Force Vector

$$[\bar{K}^e][T^e]\{\Delta^e\} = [T^e]\{F^e\}, \quad [T^e]^{-1} = [T^e]^{\mathrm{T}}$$

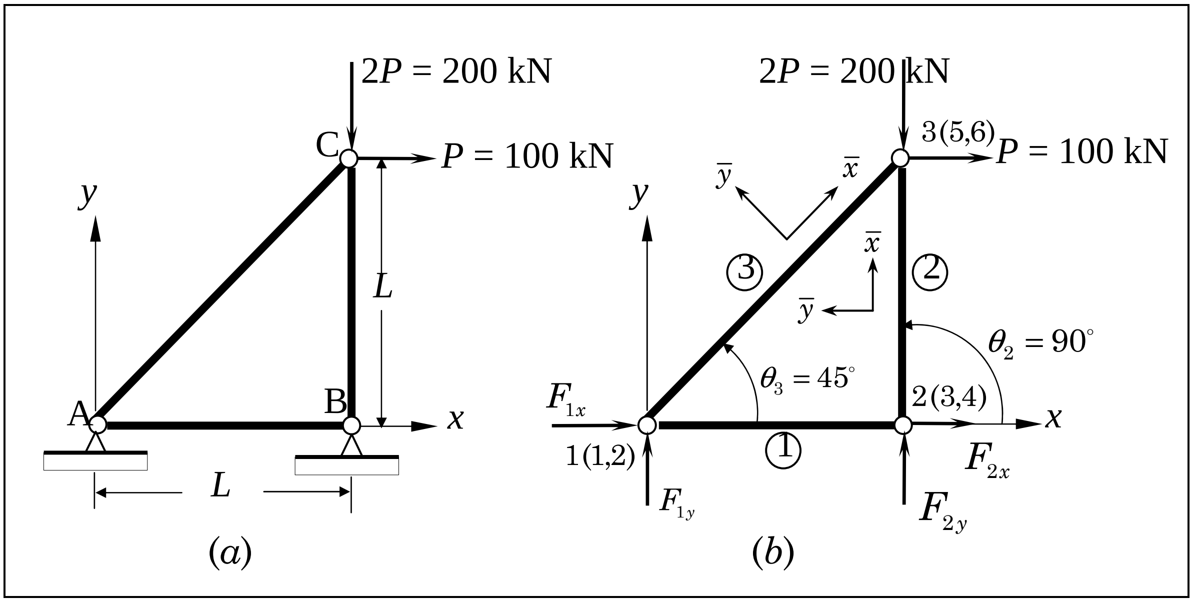
$$[T^e]^{\mathrm{T}}[\bar{K}^e][T^e]\{\Delta^e\} = \{F^e\} \ \text{ or } \ [K^e]\{\Delta^e\} = \{F^e\}$$

$$\boxed{[K^e] = [T^e]^{\mathrm{T}}[\bar{K}^e][T^e], \quad \{F^e\} = [T^e]^{\mathrm{T}}\{\bar{F}^e\}}$$

$$[K^e] = \frac{EA}{h}\begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta & -\cos^2\theta & -\frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta & -\frac{1}{2}\sin 2\theta & -\sin^2\theta \\ -\cos^2\theta & -\frac{1}{2}\sin 2\theta & \cos^2\theta & \frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & -\sin^2\theta & \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$

$$\{F^e\} = \begin{Bmatrix} F_1^e \\ F_2^e \\ F_3^e \\ F_4^e \end{Bmatrix} = \begin{Bmatrix} \bar{P}_1^e\cos\theta_e \\ \bar{P}_1^e\sin\theta_e \\ \bar{P}_2^e\cos\theta_e \\ \bar{P}_2^e\sin\theta_e \end{Bmatrix} + \begin{Bmatrix} \bar{f}_1^e\cos\theta_e \\ \bar{f}_1^e\sin\theta_e \\ \bar{f}_2^e\cos\theta_e \\ \bar{f}_2^e\sin\theta_e \end{Bmatrix}$$

<u>Example 1</u>



Example

Element number	Global nodes		Geom. prop.	Mater. prop.	Orient.
1	1	2	$A,\ h_1 = L$	E	$\theta_1 = 0°$
2	2	3	$A,\ h_2 = L$	E	$\theta_2 = 90°$
3	1	3	$A,\ h_3 = \sqrt{2}L$	E	$\theta_3 = 45°$

The element stiffness matrices are $[1/(2\sqrt{2}) = 0.3536]$

$$[K^1] = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K^2] = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K^3] = \frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ 0.3536 & 0.3536 & -0.3536 & -0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \\ -0.3536 & -0.3536 & 0.3536 & 0.3536 \end{bmatrix}$$

Assembled stiffness matrix (column 6 missing)

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
$$

$$
\left[
\begin{array}{ccccc}
K^1_{11} + K^3_{11} & K^1_{12} + K^3_{12} & K^1_{13} & K^1_{14} & K^3_{13} \\
 & K^1_{22} + K^3_{22} & K^1_{23} & K^1_{24} & K^3_{23} \\
 & & K^1_{33} + K^2_{11} & K^1_{34} + K^2_{12} & K^2_{13} \\
 & \text{symm.} & & K^1_{44} + K^2_{22} & K^2_{23} \\
 & & & & K^2_{33} + K \\
 & & & & \\
\end{array}
\right.
$$

$$
\frac{EA}{L}
\left[
\begin{array}{cccc|cc}
1.3536 & 0.3536 & -1.0 & 0.0 & -0.3536 & -0.3536 \\
 & 0.3536 & 0.0 & 0.0 & -0.3536 & -0.3536 \\
 & & 1.0 & 0.0 & 0.0 & 0.0 \\
 & & & 1.0 & 0.0 & -1.0 \\
\text{symm.} & --- & --- & --- & ---- & ---- \\
 & & & & 0.3536 & 0.3536 \\
 & & & & & 1.3536 \\
\end{array}
\right]
$$

The displacement continuity conditions are

$$
u^1_1 = u^3_1 = U_1, \quad v^1_1 = v^3_1 = V_1
$$

$$
u^1_2 = u^2_1 = U_2, \quad v^1_2 = v^2_1 = V_2
$$

$$
u^2_2 = u^3_2 = U_3, \quad v^2_2 = v^3_2 = V_3
$$

Force Balance Equations

$$F_1^1 + F_1^3 = F_x^1, \quad F_2^1 + F_2^3 = F_y^1$$
$$F_3^1 + F_1^2 = F_x^2, \quad F_4^1 + F_2^2 = F_y^2$$
$$F_3^2 + F_3^3 = F_x^3, \quad F_4^2 + F_4^3 = F_y^3$$

$$\{\Delta\} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} F_1^1 + F_1^3 \\ F_2^1 + F_2^3 \\ F_3^1 + F_1^2 \\ F_4^1 + F_2^2 \\ F_3^2 + F_3^3 \\ F_4^2 + F_4^3 \end{Bmatrix} = \begin{Bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \\ F_x^3 \\ F_y^3 \end{Bmatrix}$$

Boundary conditions

$$U_1 = V_1 = U_2 = V_2 = 0, \quad F_x^3 = P, \quad F_y^3 = -2P$$

Solution

$$\frac{EA}{L}\begin{bmatrix} 0.3536 & 0.3536 \\ 0.3536 & 1.3536 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} P \\ -2P \end{Bmatrix}$$

$$\begin{Bmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} -0.3536 & -0.3536 \\ -0.3536 & -0.3536 \\ 0.0 & 0.0 \\ 0.0 & -1.0 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \end{Bmatrix}$$

$$U_3 = (3 + 2\sqrt{2})\frac{PL}{EA} = 5.828\ \frac{PL}{EA}, \quad V_3 = -\frac{3PL}{EA}$$

$$F_x^1 = -P, \quad F_y^1 = -P, \quad F_x^2 = 0.0, \quad F_y^2 = 3P$$

Post-Computation

$$\sigma^e = -\frac{\bar{P}_1^e}{A_e} = \frac{\bar{P}_2^e}{A_e}$$

$$\begin{Bmatrix} \bar{P}_1^e \\ \bar{P}_2^e \end{Bmatrix} = \frac{A_e E_e}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1^e \\ \bar{u}_2^e \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{u}_1^e \\ \bar{v}_1^e \\ \bar{u}_2^e \\ \bar{v}_2^e \end{Bmatrix} = \begin{bmatrix} \cos\theta_e & \sin\theta_e & 0 & 0 \\ -\sin\theta_e & \cos\theta_e & 0 & 0 \\ 0 & 0 & \cos\theta_e & \sin\theta_e \\ 0 & 0 & -\sin\theta_e & \cos\theta_e \end{bmatrix} \begin{Bmatrix} u_1^e \\ v_1^e \\ u_2^e \\ v_2^e \end{Bmatrix}$$

$$u_1^1 = v_1^1 = u_2^1 = v_2^1 = 0, \quad u_1^2 = v_1^2 = u_1^3 = v_1^3 = 0$$

$$u_2^2 = u_2^3 = U_3 = (3 + 2\sqrt{2})\frac{PL}{EA}, \quad v_2^2 = v_2^3 = V_3 = -\frac{3PL}{EA}$$

$$\sigma^e = \frac{E_e}{h_e}(\bar{u}_2^e - \bar{u}_1^e)$$

$$\bar{u}_1^1 = u_1^1 \cos\theta_1 + v_1^1 \sin\theta_1 = 0; \quad \bar{u}_2^1 = u_2^1 \cos\theta_1 + v_2^1 \sin\theta_1 = 0$$

$$\bar{u}_1^2 = u_1^2 \cos\theta_2 + v_1^2 \sin\theta_2 = 0; \quad \bar{u}_1^3 = u_1^3 \cos\theta_3 + v_1^3 \sin\theta_3 = 0$$

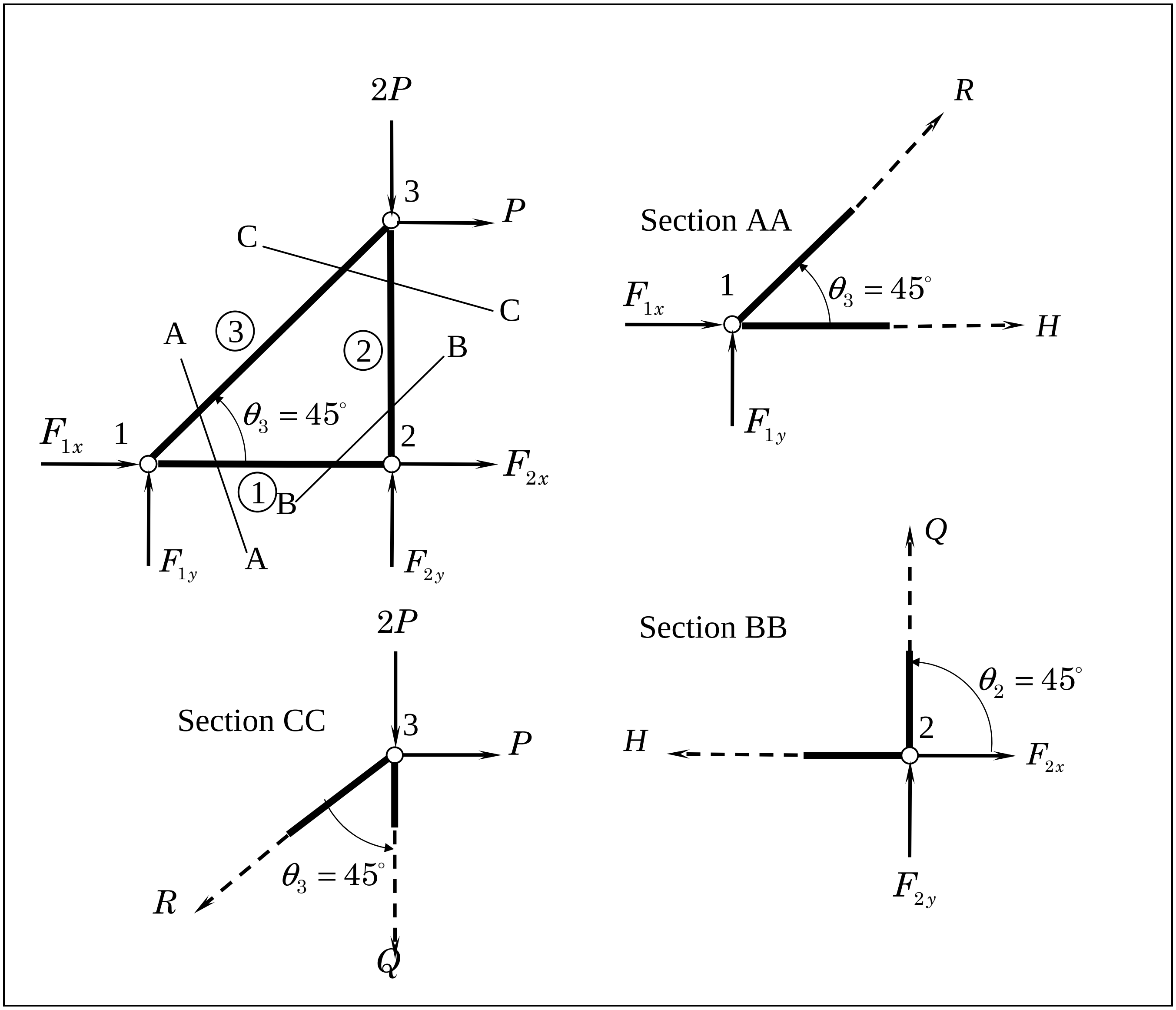
$$\bar{u}_2^2 = U_3 \cos\theta_2 + V_3 \sin\theta_2 = V_3 = -\frac{3PL}{EA}$$

$$\bar{u}_2^3 = U_3 \cos\theta_3 + V_3 \sin\theta_3 = \frac{1}{\sqrt{2}}(U_3 + V_3) = \frac{2PL}{EA}$$

$$\bar{P}_1^1 = -\bar{P}_2^1 = 0, \quad \bar{P}_1^2 = -\bar{P}_2^2 = 3P, \quad \bar{P}_1^3 = -\bar{P}_2^3 = -\sqrt{2}P$$

$$\sigma^{(1)} = 0, \quad \sigma^{(2)} = -\frac{3P}{A}, \quad \sigma^{(3)} = \sqrt{2}\,\frac{P}{A}$$

Verification of FEM Results by Comparison to Results Obtained using an Independent Approach

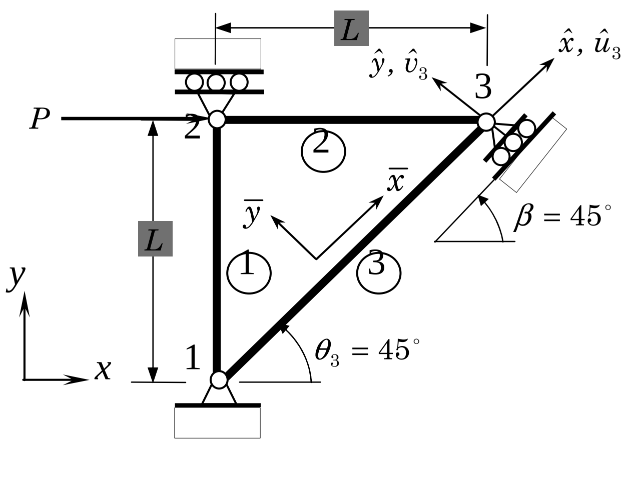


Example 2

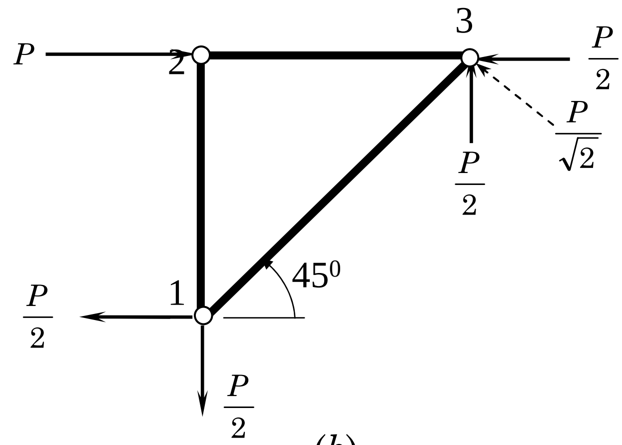
$E = 210$ GPa for all members

$L = 1$ m, $A_1 = A_2 = A_0 = 6\times10^{-4}$ m^2

$A_3 = \sqrt{2}A_0$ m^2, $P = 10^3$ kN



(a)



(b)

$$[K^1] = 10^9 \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.126 & 0.000 & -0.126 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.126 & 0.000 & 0.126 \end{bmatrix}$$

$$[K^2] = 10^9 \begin{bmatrix} 0.126 & 0.000 & -0.126 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ -0.126 & 0.000 & 0.126 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$[K^3] = 0.63 \times 10^8 \begin{bmatrix} 1.0 & 1.0 & -1.0 & -1.0 \\ 1.0 & 1.0 & -1.0 & -1.0 \\ -1.0 & -1.0 & 1.0 & 1.0 \\ -1.0 & -1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$10^8 \begin{bmatrix} 0.63 & 0.63 & 0.00 & 0.00 & -0.63 & -0.63 \\ 0.63 & 1.89 & 0.00 & -1.26 & -0.63 & -0.63 \\ 0.00 & 0.00 & 1.26 & 0.00 & -1.26 & 0.00 \\ 0.00 & -1.26 & 0.00 & 1.26 & 0.00 & 0.00 \\ 0.00 & -0.63 & -1.26 & 0.00 & 1.89 & 0.63 \\ -0.63 & -0.63 & 0.00 & 0.00 & 0.63 & 0.63 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^2 + Q_4^3 \end{Bmatrix}$$

$$u_n \equiv -u \sin\alpha + v \cos\alpha = 0 \;\rightarrow -0.7071u + 0.7071v = 0$$

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & 0 & 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \hat{u}_3 \\ \hat{v}_3 \end{Bmatrix}, \quad \beta = 45^\circ$$

$$\begin{bmatrix} 0.630 & 0.630 & 0.000 & 0.00 & -0.891 & 0.000 \\ 0.630 & 1.890 & 0.000 & -1.26 & -0.891 & 0.000 \\ 0.000 & 0.000 & 1.260 & 0.00 & -0.891 & 0.891 \\ 0.000 & -1.260 & 0.000 & 1.26 & 0.000 & 0.000 \\ -0.891 & -0.891 & -0.891 & 0.00 & 1.890 & -0.630 \\ 0.000 & 0.000 & 0.891 & 0.00 & -0.630 & 0.630 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ \hat{u}_3 \\ \hat{v}_3 \end{Bmatrix}$$

$$
= 10^{-8} \left\{ \begin{array}{c} Q_1^1 + Q_1^3 \\ Q_2^1 + Q_2^3 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ \hat{F}_{3t} \\ \hat{F}_{3n} \end{array} \right\}
$$

Solution

$U_1 = U_2 = U_4 = \hat{u}_3 = 0, \quad Q_3^1 + Q_1^2 = P = 10^6, \quad \hat{F}_{3t} = 0$

$$
10^8 \begin{bmatrix} 1.26000 & -0.89095 \\ -0.89095 & 1.88990 \end{bmatrix} \left\{ \begin{array}{c} U_3 \\ \hat{u}_3 \end{array} \right\} = \left\{ \begin{array}{c} 10^6 \\ 0 \end{array} \right\}
$$

$U_3 = 11.905 \times 10^{-3}$ m, $\quad \hat{u}_3 = 5.6122 \times 10^{-3}$ m

$F_{1x} = -500$ kN, $F_{1y} = -500$ kN, $F_{2y} = 0$ kN, $F_{3n} = 707.1$ kN