MCE 561 Computational Methods in Solid Mechanics
Nonlinear Finite Element Analysis

Many problems of engineering interest involve *nonlinear behavior*. This nonlinear phenomena generally comes from three types of sources: *nonlinear material behavior*, *large deformation theory*, and *nonlinear boundary/initial conditions*. Probably the most common source of nonlinear behavior comes from the material response. Examples of this include nonlinear elastic and plastic stress-strain behavior, and temperature dependent thermal conductivity. In addition, many solid mechanics problems involve deformations which cannot be regarded as small, and thus the strain-displacement relations must be modified. This results in a nonlinear relationship between the strains and the displacement gradients, and this type of situation is commonly referred to as *geometric nonlinearity*. Large deformations require special formulations since the undeformed and deformed elements will no longer coincide. New definitions of stress and strain are needed, and the mesh geometry must be continually updated. Boundary conditions can also lead to nonlinear formulations for cases in which the conditions change during the analysis or motion of the problem. Examples of such cases include contact problems in elasticity theory and melting or freezing problems in heat transfer. Additional information may be found in Owen and Hinton, Cook, Malkus and Plesha, Bathe, Chen and Han, and Desai and Abel.

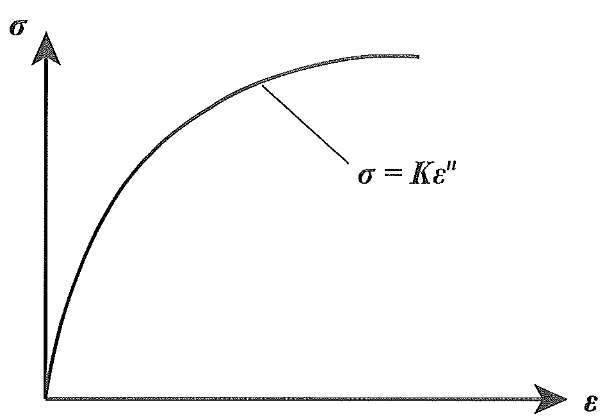
Examples of nonlinear field equations include:

One Dimensional Nonlinear/Plastic Deformation

For a bar with nonlinear material properties, the governing equations are

$$\frac{d\sigma}{dx} + f = 0$$
$$\sigma = F(\varepsilon) \quad , \quad \varepsilon = \frac{du}{dx} \tag{1}$$

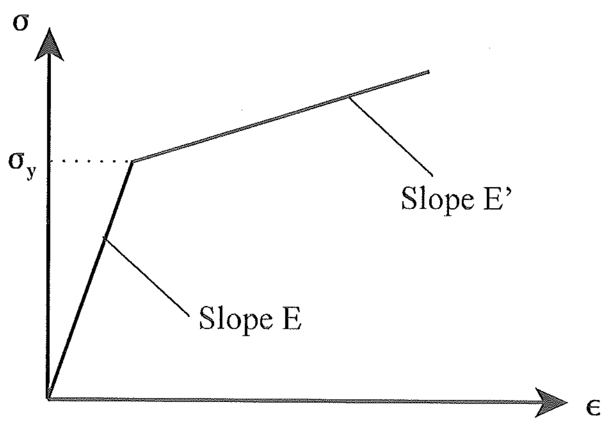
For the case of a *nonlinear elastic* material, with a stress-strain response as shown



the governing equation becomes

$$\frac{d}{dx}[K(\frac{du}{dx})^n] + f = 0 \tag{2}$$

For the case of an *elastic-plastic* response, the stress-strain plot could typically look like



For this case, a yield point σ_y, exists beyond which the material behaves differently. Unloading is usually assumed to follow the initial elastic slope, E.

Temperature Dependent Thermal Conductivity-One Dimensional Case

$$-\frac{d}{dx}[k(T)\frac{dT}{dx}] - Q = 0 \ , \quad T = T(x) \tag{3}$$

where k is the thermal conductivity which depends on the primary temperature variable, T.

Large Deflection Bending Theory of Elastic Beams

$$\begin{gathered} -\frac{d}{dx}(EA[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2]) - f \ 0 \\ \frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) - \frac{d}{dx}(EA\frac{dw}{dx}[\frac{du}{dx} + \frac{1}{2}(\frac{dw}{dx})^2]) - q = 0 \end{gathered} \tag{4}$$

where u is the longitudinal displacement, w is the transverse deflection, E is the modulus of elasticity, A is the cross-sectional area, f is the axial distributed load and q is the transverse load. If small deformations are assumed (i.e., small gradients of u and w), than this system will reduce to our previous linear cases.

Navier-Stokes Equations for Incompressible Laminar Fluid Flow

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0
\end{aligned}
\tag{5}
$$

where u and v are the x and y velocity components, ρ is the mass density and ν is the viscosity.

Solution Techniques

The solution of nonlinear problems using the finite element method is normally attempted by one of the three methods:

- *Incremental or Stepwise Procedures*
- *Iterative or Newton Methods*
- *Mixed Step-Iterative Techniques*

It is important to point out that while linear problems always had a unique answer, this is no longer the case for nonlinear situations. Consequently, even if a solution is reached by some particular incremental or iteration scheme, it may not necessarily be the solution that is sought. In order to handle this lack of uniqueness, many solution methods employ experimental data, physical reasoning and intuition to supplement particular solution techniques.

One-Dimensional Nonlinear Problems

Let us first develop the element equation for the one-dimensional nonlinear heat transfer problem with temperature dependent conductivity. The weak form for this case is given by

$$\int_{x_e}^{x_{e+1}} \left(k(T)\frac{dv}{dx}\frac{dT}{dx} - vQ\right) dx - vk(T)\frac{dT}{dx}\Big]_{x_e}^{x_{e+1}} = 0 \tag{6}$$

Using the usual Ritz-Galerkin technique

$$T = \sum_{j=1}^{N} T_j \psi_j \, , \quad v = \psi_i \tag{7}$$

in the weak form, produces the element equation

$$[K(T)]\{T\} = \{F(T)\} \tag{8}$$

where

$$K_{ij} = \int_{x_e}^{x_{e+1}} k(T) \frac{d\psi_i}{dx} \frac{d\psi_j}{dx}\, dx$$
$$F_i = \int_{x_e}^{x_{e+1}} \psi_i Q\, dx + \psi_i k(T) \frac{dT}{dx} \Big]_{x_e}^{x_{e+1}} \tag{9}$$

Thus we find for the nonlinear case that the stiffness and loading element matrices will be dependent upon the primary unknown.

Consider then the general case with an element equation of the standard form

$$[K]\{u\} = \{F\} \tag{10}$$

with $K = K(u)$. Clearly a direct solution of equation (10) through a normal inversion of the stiffness matrix is impossible, and thus incremental and/or iterative schemes are normally used to find the solution. Several such schemes have been proposed, and each method normally has particular advantages and pitfalls.

The Direct Iteration Method

This iteration method (sometimes called the *Picard* method) is based on making successive approximations to the solution by using the previous value of u to determine $K(u)$. Thus the algorithim for this technique is

$$\{u^{r+1}\} = [K(u^r)]^{-1}\{F\} \tag{11}$$

where $\{u^{r+1}\}$ is the (r+1)th approximation to the solution. If the process is convergent then the approximation $\{u^r\}$ should approach the true solution as r becomes large. This process is illustrated for a *single variable problem* in Figures 1-4 for the cases of *convex* and *concave* *K-u* relations.

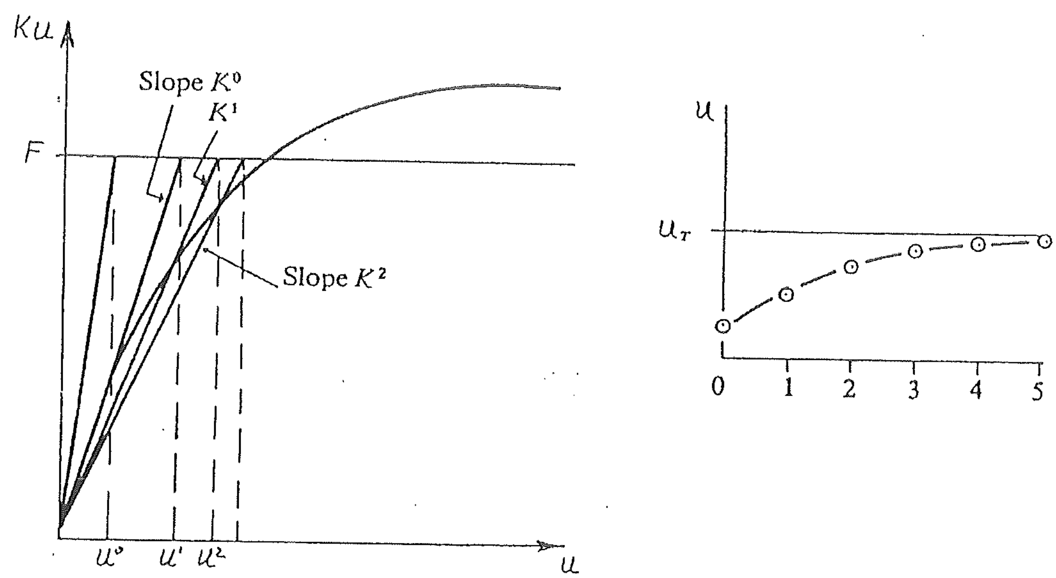


Figure 1. Direct Iteration Method, Convex K-u Relation, Low Initial Solution

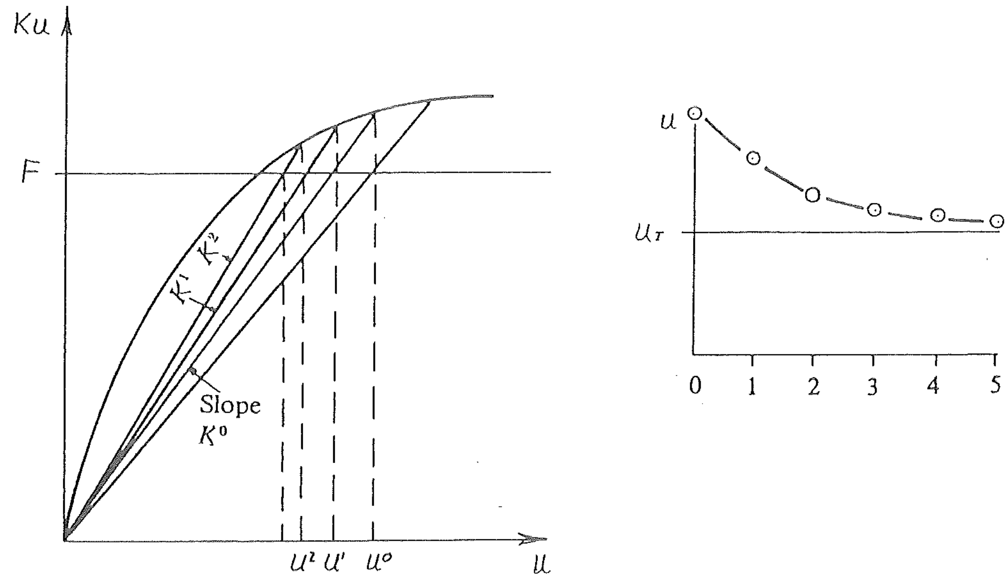


Figure 2. Direct Iteration Method, Convex K-u Relation, High Initial Solution

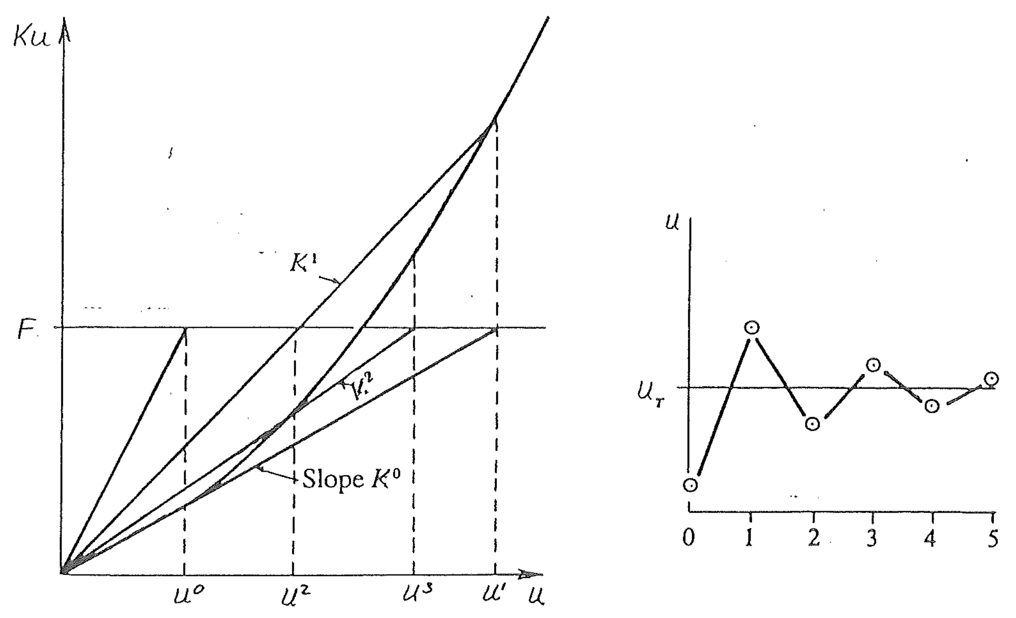


Figure 3. Direct Iteration Method, Concave K-u Relation, Low Initial Solution

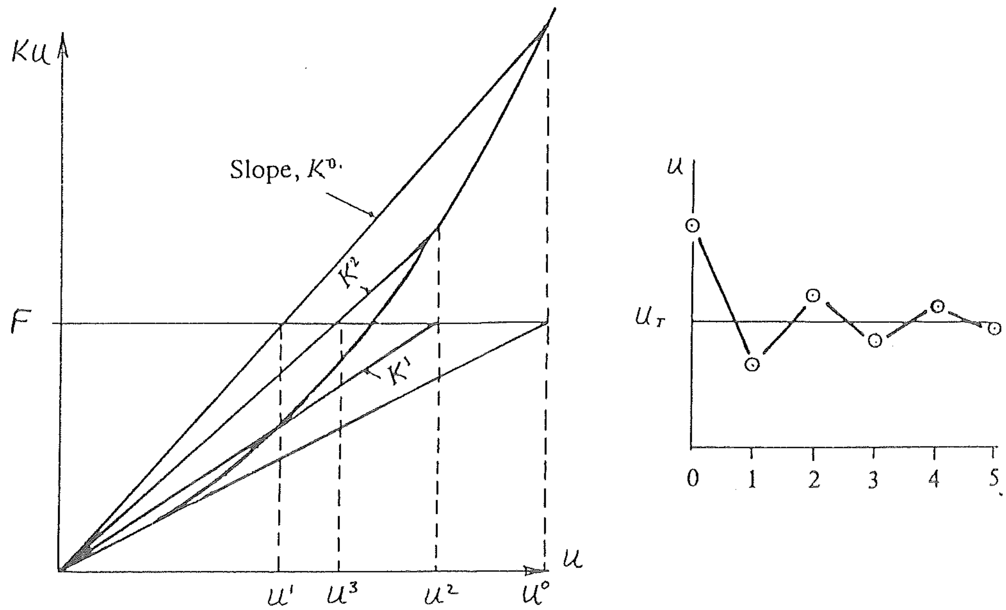


Figure 4. Direct Iteration Method, Concave K-u Relation, High Initial Solution

For the case of a concave K-u relation, oscillations and divergence can occur, and thus this scheme *cannot guarantee general convergence.* Another problem with the direct iteration scheme is that *it is necessary to recalculate the inverse of the new stiffness matrix for each iteration.* This situation is very computationally demanding for problems with large numbers of degrees of fredom.

The Newton-Raphson Method

During any typical iteration step, the approximate solution to the problem defined by equation (10) will not exactly satisfy the equation. Such approximate solutions will lead to a set of *residual forces* defined by

$$\{R\} = [K]\{u\} - \{F\} \tag{12}$$

These residual forces can be thought of as a measure of the departure from equilibrium or balance, and they are clearly dependent upon the primary variable u.

If $\{u^r\}$ is an approximate solution to relation (10), then the true solution at $u_j^r + \Delta u_j^r$ can be written using the *truncated Taylor expansion*

$$R_i(u_j^{r+1}) = R_i(u_j^r) + (\frac{\partial R_i}{\partial u_j})^r \Delta u_j^r = 0 \ , \quad \text{where} \quad u_j^{r+1} = u_j^r + \Delta u^r \tag{13}$$

and therefore we can write

$$\{R^r\} = -[J(u^r)]\{\Delta u^r\} \tag{14}$$

where the term [J] is referred to as the *Jacobian matrix* given by

$$J_{ij}(u^r) = (\frac{\partial R_i}{\partial u_j})^r = K_{ij}(u^r) + \sum_{k=1}^{N} (\frac{\partial K_{ik}}{\partial u_j})^r u_k^r \tag{15}$$

Note that the last term in equation (15) is not symmetric. If this term is neglected in order to maintain symmetry, relation (14) gives

$$\{R(u^r)\} = -[K(u^r)]\{\Delta u^r\} \tag{16}$$

or since $\Delta u_r = u_{r+1} - u_r$,

$$[K(u^r)]\{u^{r+1}\} = \{F\} \tag{17}$$

which is identical to relation (11) which governs the direct iteration method. Consequently, the non-symmetric terms in (15) must be retained in order to construct a different (and hopefully better) iteration method.

Retaining both terms in equation (15), we can write

$$\boldsymbol{J}(u) = \boldsymbol{K}(u) + \boldsymbol{K}'(u)\, \boldsymbol{u} \tag{18}$$

and therefore the increment or correction in the vector of unknowns $\{u^r\}$ is given by

$$\{\Delta u^r\} = -[J(u^r)]^{-1}\{R(u^r)\} = -([K(u^r)] + [K'(u^r)]\{u\})^{-1}\{R(u^r)\} \tag{19}$$

which is the statement of the *Newton-Raphson method.* This approach allows the correction to the vector of unknows to be obtained from the residual force vector for any iteration, and the process is continued until convergence has been acheived. Such convergence is usually measured by the magnitudes of the residual vector components with the idea that these components should become small as the iteration process continues. Thus the Newton-Raphson method seeks to reduce the load imbalance and the correction to zero. Again this scheme is illustrated for the single variable problem in Figures 5-8 for the case of convex and concave K-u relations. It should be apparent from the figures that this scheme can be thought of as a *tangential stiffness method* in that over each iteration the problem is linearized and the current tangential stiffness is used to predict the response. It should also be obvious that the Newton-Raphson method has a higher convergence rate than that of the direct iteration method. However, this scheme requires that the Jacobian matrix (tangential stiffness matrix) be evaluated and inverted at each iteration step which as mentioned previously, can be computationally demanding for large problems. Also this scheme will have trouble in handling *perfectly plastic* or *strain-softening materials* in which the tangent stiffness may become zero, thus yielding a singular or ill-conditioned tangential stiffness matrix.

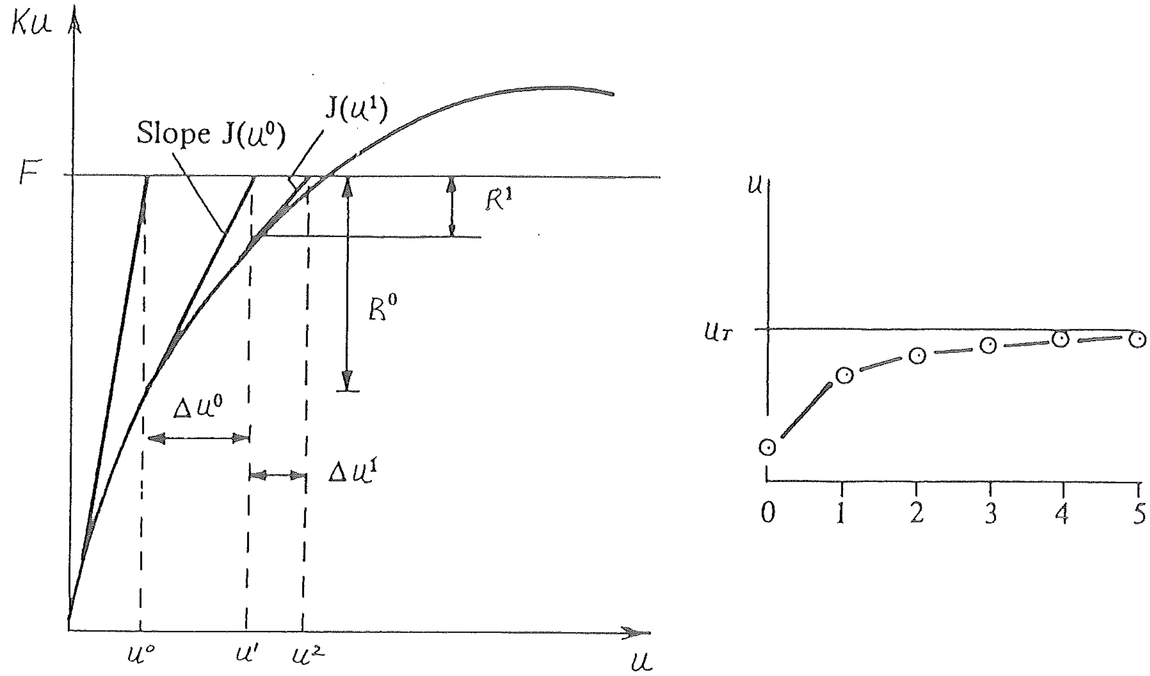


Figure 5. Newton-Raphson Method, Convex K-u Relation, Low Initial Solution

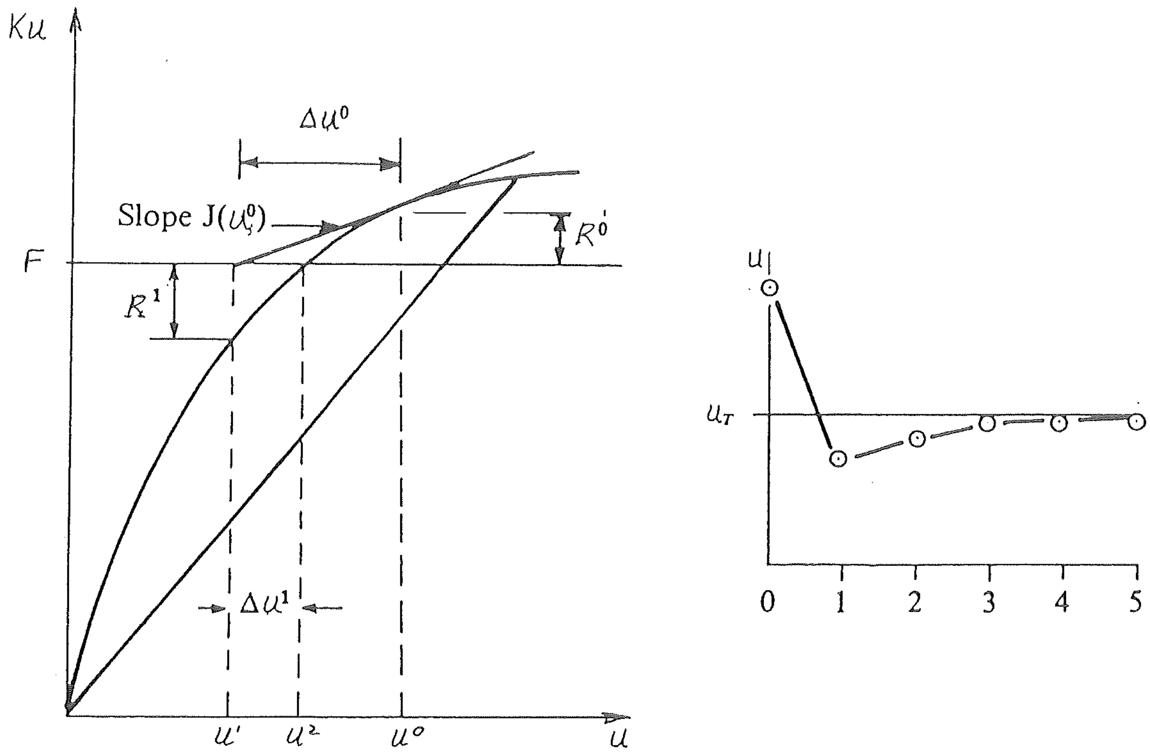


Figure 6. Newton-Raphson Method, Convex K-u Relation, High Initial Solution

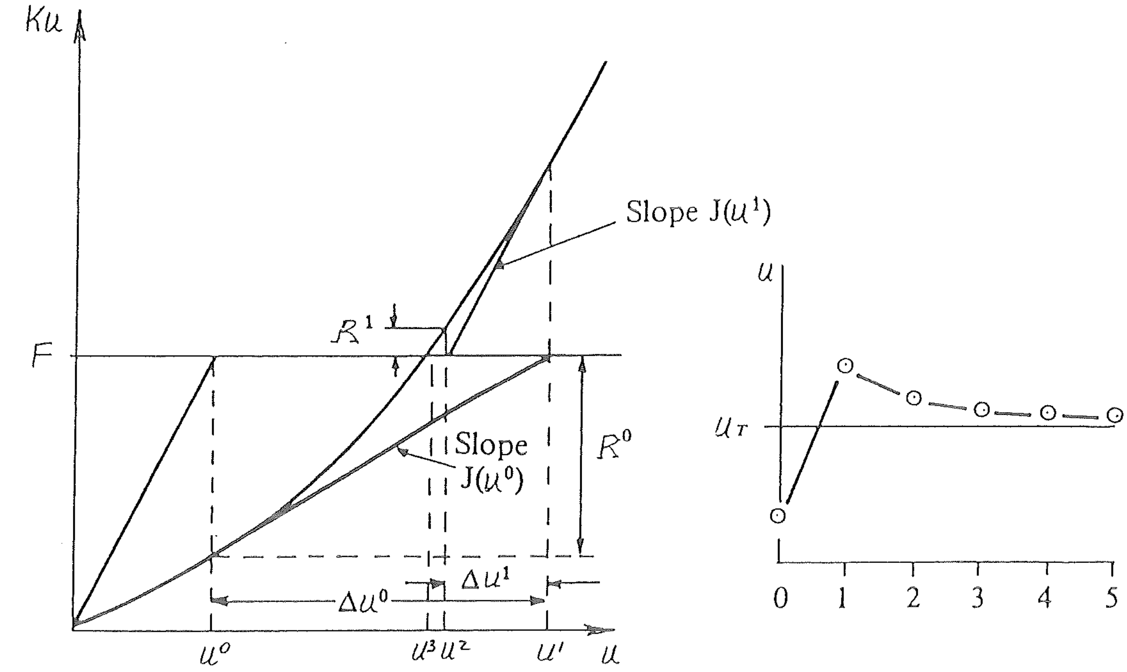


Figure 7. Newton-Raphson Method, Concave K-u Relation, Low Initial Solution

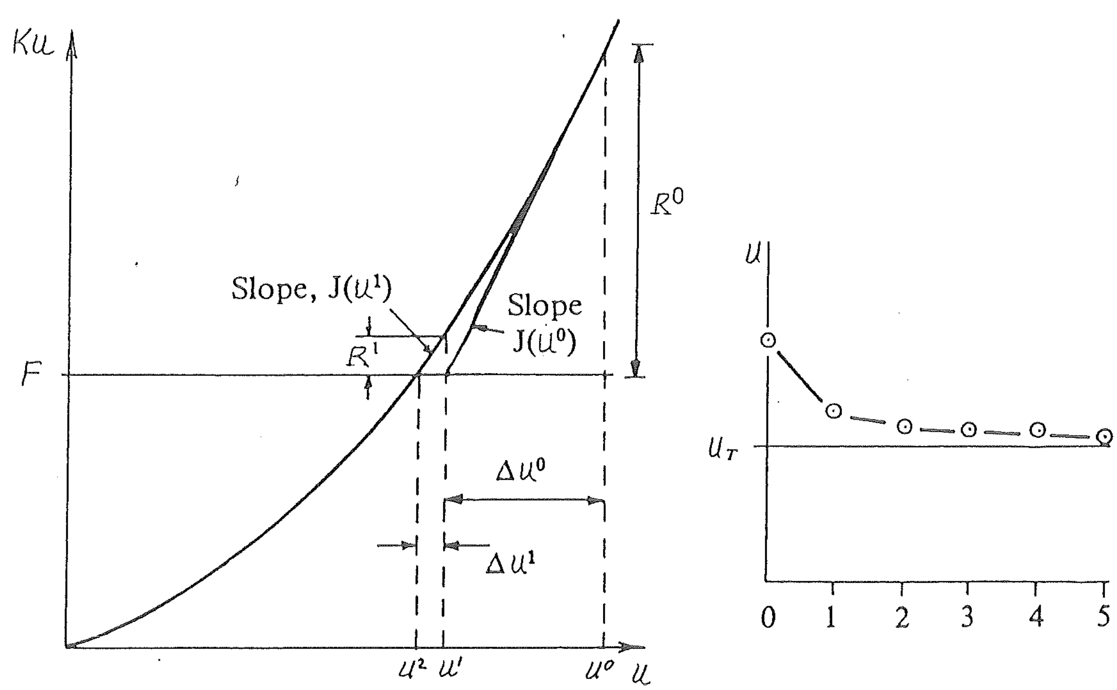


Figure 8. Newton-Raphson Method, Concave K-u Relation, High Initial Solution

The Modified Newton-Raphson (Initial Stiffness) Method

The previous methods required for each iteration, the evaluation and inversion of the entire set of equations describing the discretized problem. One of the modifications of the Newton-Raphson method is to replace the tangential stiffness matrix at all iteration steps, by the stiffness corresponding to the *initial* trial value of **u**. Thus the inversion of tangential stiffness matrix is carried out only for the first iteration, and all subsequent iterations retain this same stiffness matrix. Therefore this iteration scheme can be written as

$$\{\Delta u^r\} = -[J(u^0)]^{-1}\{R(u^r)\} \tag{20}$$

Because the same stiffness matrix $\mathbf{J}(u^0)$ is used at each iteration step, the computational effort is greatly reduced; however, the convergence rate is also reduced as can be seen in the single variable example shown in Figure 9. The iterative algorithim for this case is identical to the Newton-Raphson method of the previous section. The method has been shown to be unconditonally convergent, and can even be employed in situations where the material exhibits negative stiffness.

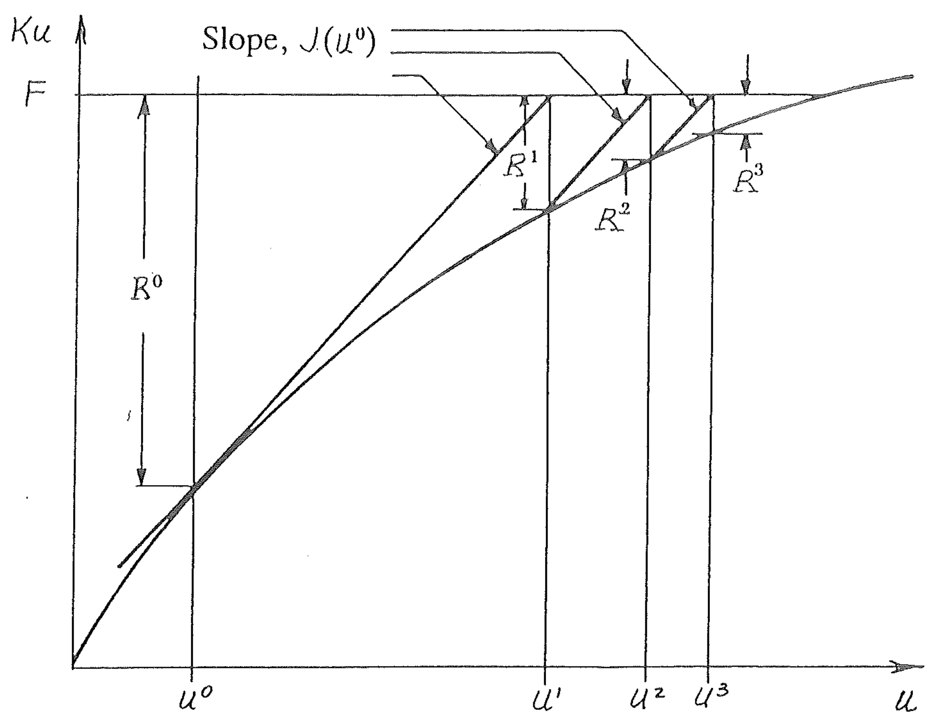


Figure 9. Modified Newton-Raphson (Initial Stiffness) Method

Additional variations of the modified Newton-Raphson method could include periodic updating of the stiffness matrix at selected iterations to improve the convergence rate. The relative economies of the modified versus the original Newton-Raphson method will depend to a large extent on the type and degree of nonlinearity in the problem under study.

Incremental Methods

Incremental or stepwise procedures subdivide the load into many small segments or increments. The loading is thus applied one increment at a time, and during the application of each increment the system is assumed to obey a linear relation. Thus over each increment a fixed value of the stiffness is used, but this stiffness may take different values during different load increments. The solution for each loading step is therefore obtained as an increment to be added to the accumulated total solution. The incremental procedure thus approximates the nonlinear problem as a series of linear problems, and thus the nonlinearity in treated as *piecewise linear*.

The method can be developed by writing equation (10) in the form

$$[K(u)]\{u\} = \lambda\{F_o\} \tag{21}$$

where λ is a scalar loading parameter. Differentiating with respect to λ gives

$$[J(u)]\frac{d\{u\}}{d\lambda} = \{F_o\}\ , \quad \text{or} \quad \frac{d\{u\}}{d\lambda} = [J(u)]^{-1}\{F_o\} \tag{22}$$

where J(u) is the Jacobian or tangential stiffness matrix defined previously.

Equation (22) can be numerically integrated using any one of several standard schemes. One simple procedure is to use the *Euler method*, which proposes the approximate solution as

$$\{u^{r+1}\} - \{u^r\} = [J(u^r)]^{-1}\{F_o\}\Delta\lambda_r = [J(u^r)]^{-1}\{\Delta F_r\} \tag{23}$$

where the subscripts refer to the increments of λ and F. Figure 10 illustrates this algorithim using the Euler method as proposed in equation (23). It can be seen that this scheme will produce an approximate solution that drifts further and further from the exact solution with each incremental step.

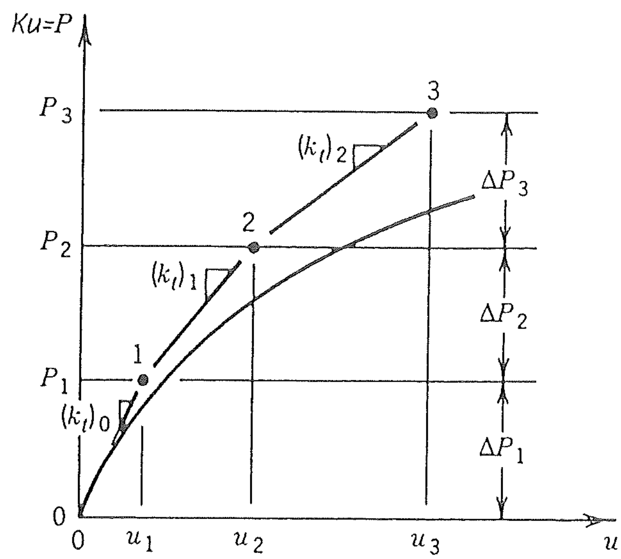


Figure 10. Incremental Method Using Simple Euler Scheme

Improved integration schemes, such as *corrected Euler*, *Runge-Kutta*, or *predictor-corrector*, can improve this short-coming. Also this drift from the true solution can be reduced by introducing the *load imbalance* as a corrective term. Such methods are sometimes called *incremental with a one-step Newton-Raphson correction*. Figure 11 shows the improvement using such a scheme.

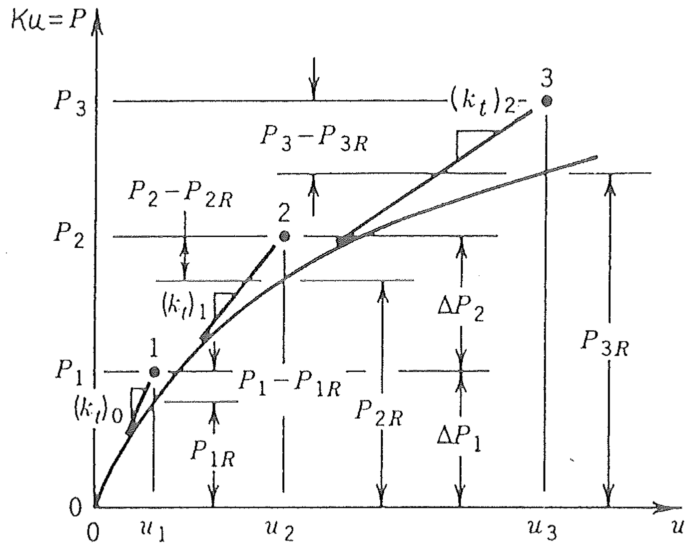


Figure 11. Incremental Method with Load Corrections

$$u^{r+1} = u^{r} + J^{-1}\,[\Delta F_{r} + R^{r}]$$