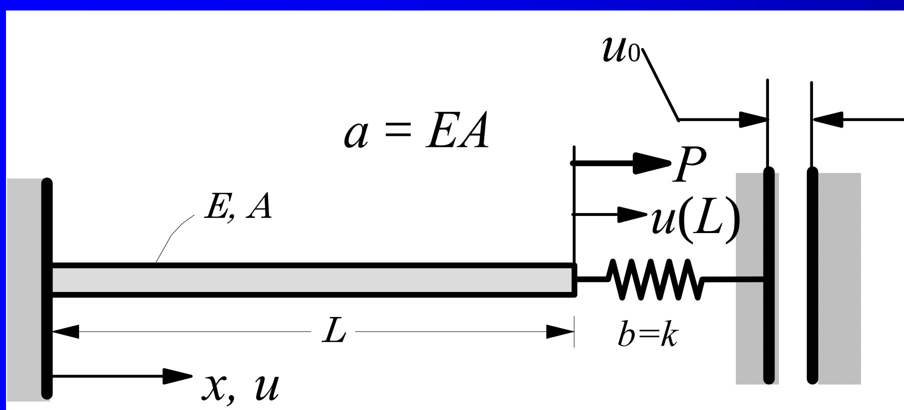
FINITE ELEMENT ANALYSIS OF A 1D MODEL PROBLEM WITH A SINGLE VARIABLE

Finite element model development of a linear 1D model differential equation involving a single dependent unknown (governing equations, FE model development weak form).

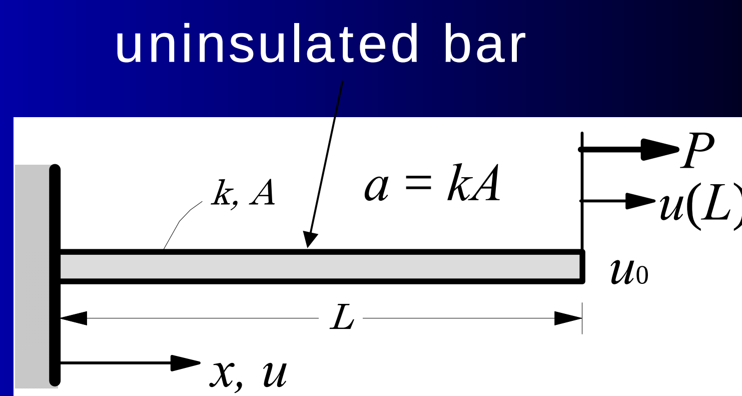
GOVERNING EQUATION

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u = f(x) \ \text{ in } \ \Omega=(0,L)$$

$$a\frac{du}{dx}+b(u-u_0)=P \ \text{ at a boundary point}$$



Elastic deformation of a bar



Heat transfer in a bar

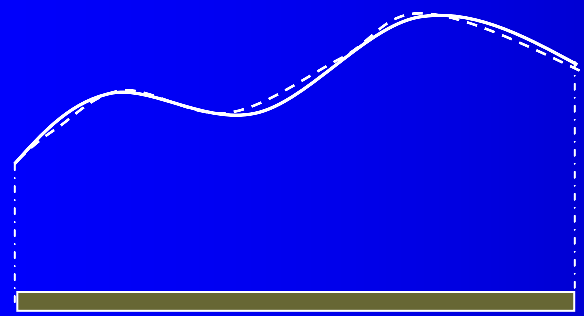
$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu-f=0$$

Table 3.2.1 Some examples of engineering problems governed by the second-order equation (3.2.1) (see the footnote for the meaning of some parameters*).

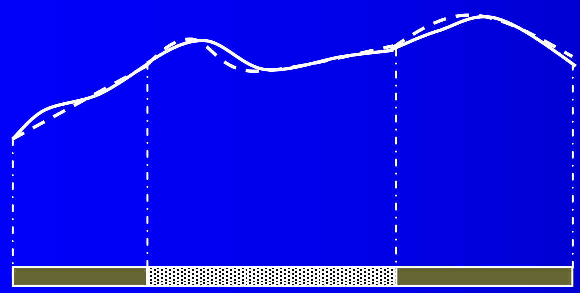
Field of study	Primary variable u	Problem data a	Problem data c	Problem data f	Secondary variable Q
Heat transfer	Temperature $T - T_\infty$	Thermal conductance kA	Surface convection $AP\beta$	Heat generation f	Heat Q
Flow through porous medium	Fluid head ϕ	Permeability μ	0	Infiltraction f	Point source Q
Flow through pipes	Pressure P	Pipe resistance $1/R$	0	0	Point source Q
Flow of viscous fluids	Velocity v_x	Viscosity μ	0	Pressure gradient $-dP/dx$	Shear stress σ_{xz}
Elastic cables	Displacement u	Tension T	0	Transverse force f	Point force P
Elastic bars	Displacement u	Axial stiffness EA	0	Axial force f	Point force P
Torsion of bars	Angle of twist θ	Shear stiffness GJ	0	0	Torque T
Electrostatics	Electrical potential ϕ	Dielectric constant ϵ	0	Charge density ρ	Electric flux E

*k = thermal conductance; β = convective film conductance; p = perimeter; P = pressure or force; T_∞ = ambient temperature of the surrounding fluid medium; $R = 128\mu h/(\pi d^4)$ with μ being the viscosity, h the length, and d the diameter of the pipe; E = Young's modulus; A = area of cross section; J = polar moment of inertia.

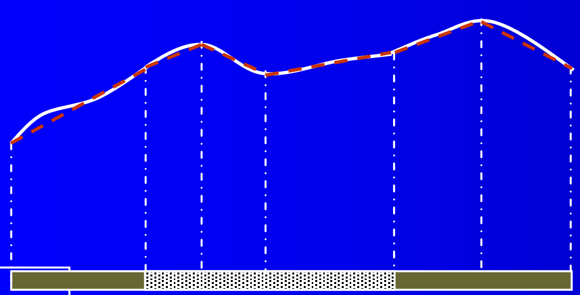
FINITE ELEMENT APPROXIMATION
(NEED FOR SEEKING SOLUTION ON SUB-INTERVALS)

1. 

Approximation of the actual solution over the entire domain requires higher-order approximation

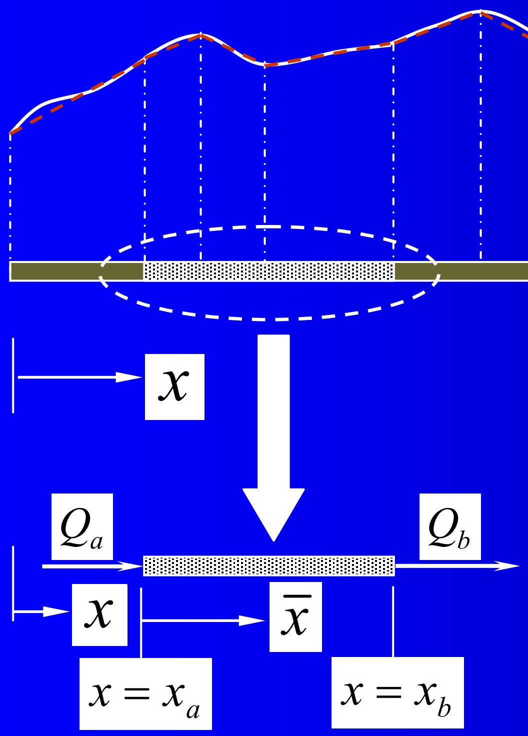
2. 

Actual solution may be defined by Sub-intervals because of discontiuity of the data $a(x)$.



Approximation over sub-intervals subintervals allows lower-order Approximation of the actual solution

FINITE ELEMENT DISCRETIZATION



Approximation over sub-intervals subintervals allows lower-order Approximation of the actual solution

A typical element (geometry and 'forces')

Q_a, Q_b end forces or heats

$h = x_b - x_a$ = element length

WEAK FORM OVER AN ELEMENT

$$0=\int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+c(x)u-f(x)\right]dx$$

$$=\int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right]dx-\left[w\cdot a\frac{du}{dx}\right]_{x_a}^{x_b}$$

$$=\int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right]dx-w(x_a)\cdot\left(-a\frac{du}{dx}\right)_{x_a}-w(x_b)\cdot\left(a\frac{du}{dx}\right)_{x_b}$$

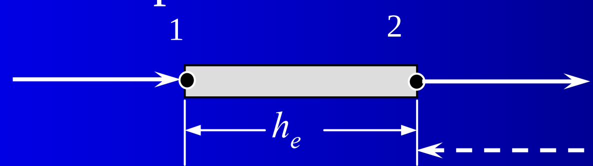
$$=\int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx}+cwu-wf\right]dx-w(x_a)Q_{\mathrm{a}}-w(x_b)\cdot Q_b$$

Heat/Force input

$$\left(-a\frac{du}{dx}\right)_{x_a}\equiv Q_a$$

Force out

$$\left(a\frac{du}{dx}\right)_{x_b}\equiv Q_b$$



Heat input

LINEAR AND BILINEAR FORMS AND THE VARIATIONAL PROBLEM

Weak Form

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu - wf\right]dx - w(x_a)Q_{\mathrm{a}} - w(x_b)\cdot Q_b \\
&= \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu\right]dx - \left[\int_{x_b}^{x_a} wf\,dx + w(x_a)Q_{\mathrm{a}} + w(x_b)\cdot Q_b\right] \\
&= B(w,u) - l(w)
\end{aligned}
$$

Variational Problem: Find u such that

$$
B(w,u) = l(w) \quad \text{holds for all} \quad w
$$

Bilinear Form and Linear Form

$$
B(w,u) = \int_{x_a}^{x_b}\left[a\frac{dw}{dx}\frac{du}{dx} + cwu\right]dx, \quad l(w) = \left[\int_{x_b}^{x_a} wf\,dx + w(x_a)Q_{\mathrm{a}} + w(x_b)\cdot Q_b\right]
$$

EQUIVALENCE BETWEEN MINIMUM OF A QUADRATIC FUNCTIONAL AND WEAK FORM

Quadratic Functional:

$$I(u) \equiv \tfrac{1}{2} B(u,u) - l(u)$$

$$= \overbrace{\frac{1}{2}\int_{x_a}^{x_b}\left[a\left(\frac{du}{dx}\right)^2 + cu^2\right]dx}^{\text{Strain energy}} - \overbrace{\left[\int_{x_b}^{x_a} uf\,dx + u(x_a)Q_{\mathrm{a}} + u(x_b)\cdot Q_b\right]}^{\text{Work done by applied forces}}$$

Variational Problem: Find u such that $I(u)$ is a minimum:

$$\delta I = 0 \Rightarrow B(\delta u, u) - l(\delta u) = 0 \quad \text{for all } \delta u$$

which is the same as the weak form or the variational problem with $\delta u = w$

FINITE ELEMENT MODEL

Finite element approximation (to be derived later)

$$u(x) \approx U^e(x) = \sum_{j=1}^{n} u_j^e \psi_j^e(x)$$

Finite element model

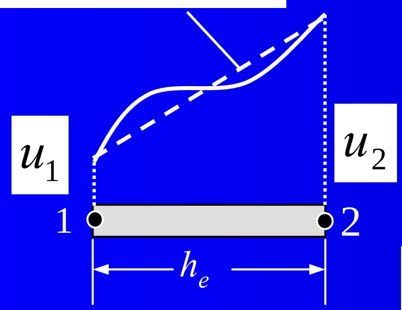
$$[K^e]\{u^e\} = \{F^e\}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} + c\psi_i \psi_j \right) dx,$$

$$F_i^e = \int_{x_a}^{x_b} f\psi_i \; dx + \psi_i(x_a)\, Q_a + \psi_i(x_b)\, Q_b$$

APPROXIMATION FUNCTIONS FOR LINEAR ELEMENT

$$U(x) = c_1 + c_2 x$$



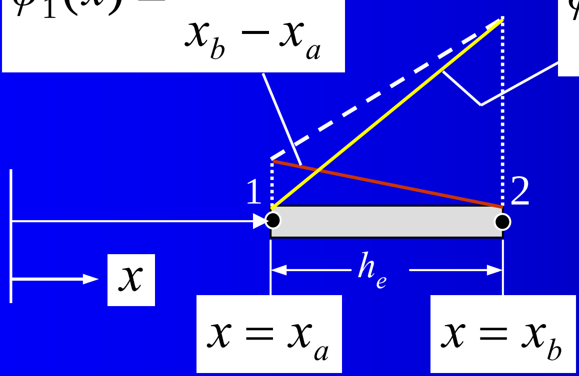
$$u(x) \approx U(x) = c_1 + c_2 x$$

$$U(x_b) \equiv u_b = u_2 \qquad U(x_a) \equiv u_a = u_1$$

$$U(x_a) = u_1 = c_1 + c_2 x_a$$
$$U(x_b) = u_2 = c_1 + c_2 x_b$$

$$\rightarrow c_1 = \frac{u_1 x_b - u_2 x_a}{x_b - x_a}, \quad c_2 = \frac{u_2 - u_1}{x_b - x_a}$$

$$\psi_1(x) \equiv \frac{x_b - x}{x_b - x_a} \qquad \psi_2(x) \equiv \frac{x - x_a}{x_b - x_a}$$



$$u(x) \approx U(x) = c_1 + c_2 x$$
$$= \frac{u_1 x_b - u_2 x_a}{x_b - x_a} + \frac{u_2 - u_1}{x_b - x_a} x$$
$$= \psi_1(x) u_1 + \psi_2(x) u_2$$

ALTERNATE DERIVATION OF APPROXIMATION FUNCTIONS

(Linear Element)

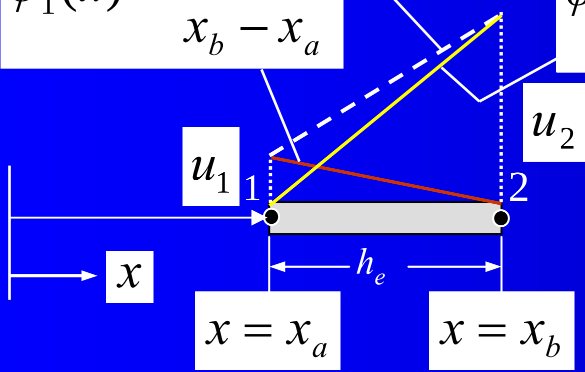
$$\psi_i(x_j) \equiv \delta_{ij} = \begin{cases} 0, \text{ if } & i \neq j \\ 1, \text{ if } & i = j \end{cases}$$

$x_1 \equiv x_a, \; x_2 \equiv x_b$ (interpolation functions)

$$U(x) = \psi_1(x)u_1 + \psi_2(x)u_2$$

$$\psi_1(x) \equiv \frac{x_b - x}{x_b - x_a}$$

$$\psi_2(x) \equiv \frac{x - x_a}{x_b - x_a}$$



Alternate derivation using the interpolation property

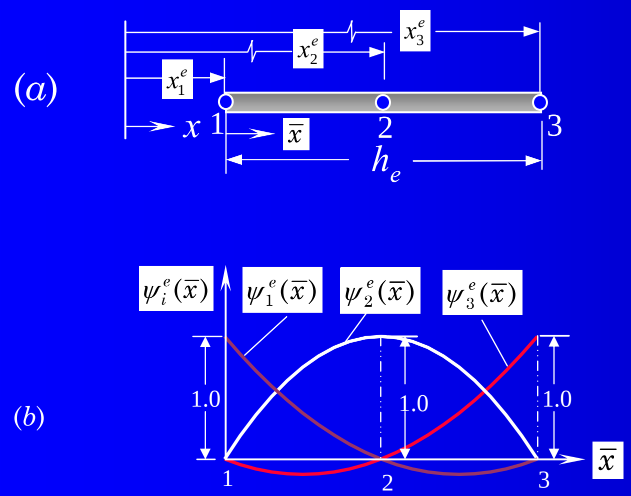
$$\psi_1(x) = \alpha_1(x_b - x)$$

$$\text{and } \psi_1(x_a) = 1 \rightarrow \alpha_1 = (x_b - x_a)^{-1}$$

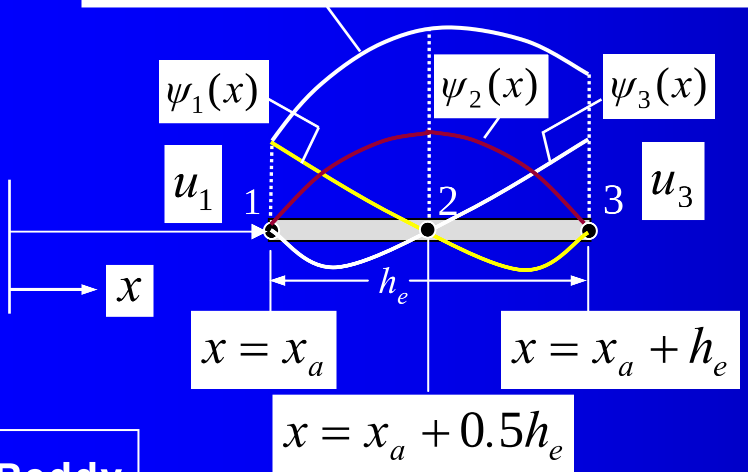
$$\psi_2(x) = \alpha_2(x - x_a)$$

$$\text{and } \psi_2(x_b) = 1 \rightarrow \alpha_2 = (x_b - x_a)^{-1}$$

ALTERNATE DERIVATION OF APPROXIMATION FUNCTIONS
(Quadratic Element)



$$U(x) = \psi_1(x)u_1 + \psi_2(x)u_2 + \psi_3(x)u_3$$



Alternate derivation using the interpolation property

$$\psi_1(x) = \alpha_1(x_a + h - x)(x_a + 0.5h - x)$$

$$\text{and } \psi_1(x_a) = 1 \rightarrow \alpha_1 = \frac{2}{h^2}$$

$$\psi_2(x) = \alpha_2(x - x_a)(x_a + h - x)$$

$$\text{and } \psi_2(x_a + 0.5h) = 1 \rightarrow \alpha_2 = \frac{4}{h^2}$$

$$\psi_3(x) = \alpha_3(x - x_a)(x_a + 0.5h - x)$$

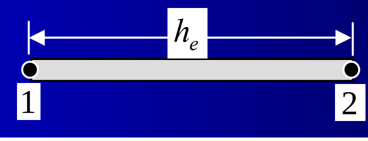
$$\text{and } \psi_3(x_a + h) = 1 \rightarrow \alpha_3 = -\frac{2}{h^2}$$

NUMERICAL EVALUATION OF COEFFICIENTS
for element-wise constant data

For constant *data* : $\quad a = a_e,\ c = c_e,\ f = f_e$

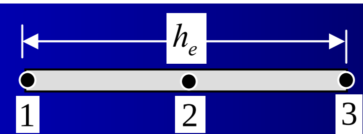
$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + c\psi_i\psi_j \right) dx, \quad F_i^e = \int_{x_a}^{x_b} f\psi_i \ dx + Q_i$$

Linear element:



$$[K^e] = \frac{a_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

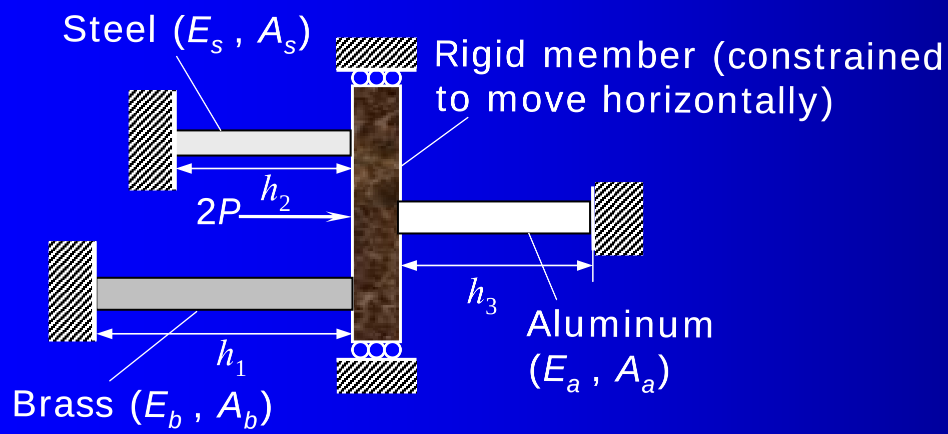
Quadratic element:



$$[K^e] = \frac{a_e}{3h_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{c_e h_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad \{F^e\} = \frac{f_e h_e}{2}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

NUMERICAL EXAMPLE - 1

Problem: Wish to determine the deformation and stresses in the three members of the structure.



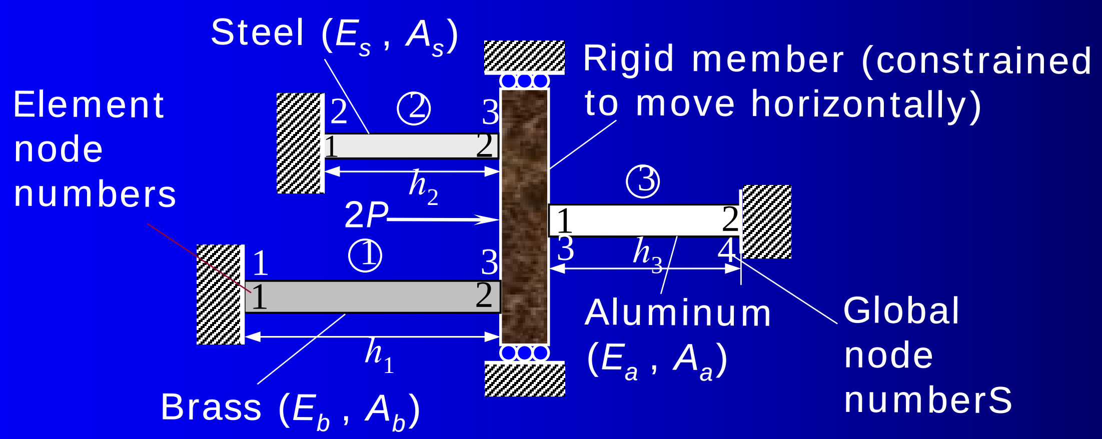
$h_1 = 8\,\text{ft.},\ h_2 = 4\,\text{ft},\ h_3 = 6\,\text{ft.},$
$d_s = 0.5\,\text{in.},\ d_b = 0.875\,\text{in.},$
$d_a = 1.0\,\text{in.},\ E_s = 30\times 10^6\,\text{psi},$
$E_b = 14\times 10^6\,\text{psi},\ E_a = 10\times 10^6\,\text{psi},$
$P = 3000\,\text{lb}$

Solution: (1) Note that the governing equation is

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \text{ in each element}$$

NUMERICAL EXAMPLE – 1 (continued)

(2) We use linear elements to represent the members of the structure and label the elements, element nodes and global nodes.



(3) The element equations for a typical element are

$(a_e = E_e A_e ,\, c_e = 0,\, f_e = 0)$

$$\frac{E_e A_e}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

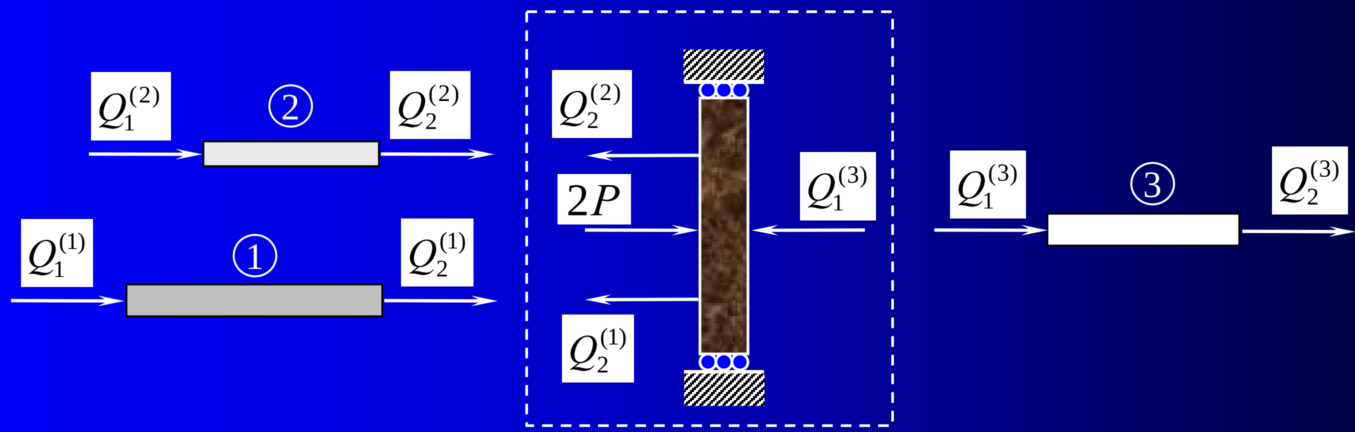
NUMERICAL EXAMPLE – 1 (continued)

(4) The displacement boundary conditions expressed in terms of the global displacements are

$$U_1 = 0,\ U_2 = 0,\ U_4 = 0$$

(5) The force equilibrium conditions are obtained by looking at the equilibrium of forces on the rigid bar. We have

$$-Q_2^{(1)} + 2P - Q_2^{(2)} - Q_1^{(3)} = 0 \quad \rightarrow \quad Q_2^{(1)} + Q_2^{(2)} + Q_1^{(3)} = 2P$$



NUMERICAL EXAMPLE – 1 (continued)

(6) The equilibrium conditions suggest that we must add the second equation of element 1, the second equation of element 2, and the first equation of element 3 so that we can replace the sum of the *Q*'s with 2*P*. The element equations in terms of the global displacements are

$$\frac{E_1A_1}{h_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^1 = \cancel{U_1}^{\,0} \\ u_2^1 = U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix} \qquad \frac{E_2A_2}{h_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^2 = \cancel{U_2}^{\,0} \\ u_2^2 = U_3 \end{Bmatrix} = \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

$$\frac{E_3A_3}{h_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^3 = U_3 \\ u_2^3 = \cancel{U_4}^{\,0} \end{Bmatrix} = \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

Using the fact that $U_1 = U_2 = U_4 = 0$, we obtain

$$\left(\frac{E_1A_1}{h_1} + \frac{E_2A_2}{h_2} + \frac{E_3A_3}{h_3}\right)U_3 = Q_2^1 + Q_2^2 + Q_1^3 = 2P$$

NUMERICAL EXAMPLE – 1 (continued)

Thus we can compute U_3 , which represents the elongation in elements 1 and 2 and compression in element 3.

(7) We can then determine the member forces $Q_2^{(e)}$ using

$$Q_2^{(1)} = \frac{E_1 A_1}{h_1} U_3, \; Q_2^{(2)} = \frac{E_2 A_2}{h_2} U_3, \; Q_2^{(3)} = -\frac{E_3 A_3}{h_3} U_3$$

The member stresses are then can be computed from

$$\sigma^{(1)} = \frac{Q_2^{(1)}}{A_1} = \frac{E_1}{h_1} U_3, \; \sigma^{(2)} = \frac{Q_2^{(2)}}{A_2} = \frac{E_2}{h_2} U_3, \; \sigma^{(3)} \frac{Q_2^{(3)}}{A_3} = -\frac{E_3}{h_3} U_3$$

This completes Example 1 (one can substitute the given data to obtain numerical values of the displacements, forces, and stresses in the members).

NUMERICAL EXAMPLE – 2

Problem: Wish to determine the numerical solution of the differential equation

$$-\frac{d^2u}{dx^2} - u = -x^2 \ \text{ in } \ 0 < x < 1$$
$$u(0) = 0, \quad u(1) = 0$$

Solution: We have the following correspondence compared to the model equation:

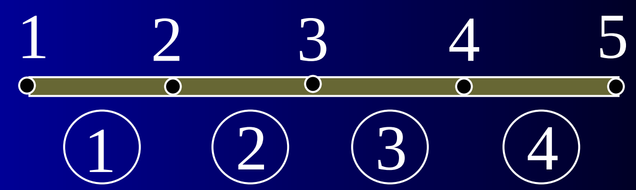
$$a = 1, \ c = -1, \ f = -x^2, \ f_i^e = -\int_{x_a}^{x_b} x^2 \psi_i \ dx$$

(1) We wish to use a mesh of linear elements to solve the problem. The equations of a typical element are

$$\left(\frac{1}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} u_1^e \\ u_2^e \end{Bmatrix} = -\frac{1}{h_e} \begin{Bmatrix} \frac{x_b}{3}(x_b^3 - x_a^3) - \frac{1}{4}(x_b^4 - x_a^4) \\ -\frac{x_a}{3}(x_b^3 - x_a^3) + \frac{1}{4}(x_b^4 - x_a^4) \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix}$$

NUMERICAL EXAMPLE – 2 (continued)

(2) We consider a mesh of 4 linear elements (h = 0.25). The element equations are

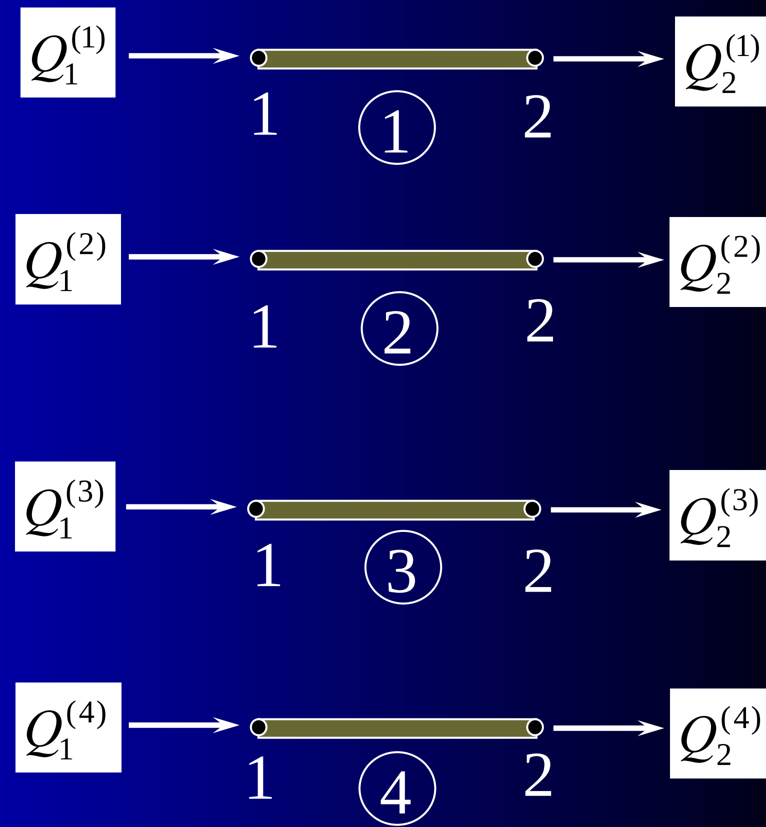


$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^1 \\ u_2^1 \end{Bmatrix} = -\begin{Bmatrix} 0.00130 \\ 0.00391 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^2 \\ u_2^2 \end{Bmatrix} = -\begin{Bmatrix} 0.01432 \\ 0.02232 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^3 \\ u_2^3 \end{Bmatrix} = -\begin{Bmatrix} 0.04297 \\ 0.05599 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

$$\frac{1}{24}\begin{bmatrix} 94 & -97 \\ -97 & 94 \end{bmatrix}\begin{Bmatrix} u_1^4 \\ u_2^4 \end{Bmatrix} = -\begin{Bmatrix} 0.08724 \\ 0.10547 \end{Bmatrix} + \begin{Bmatrix} Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

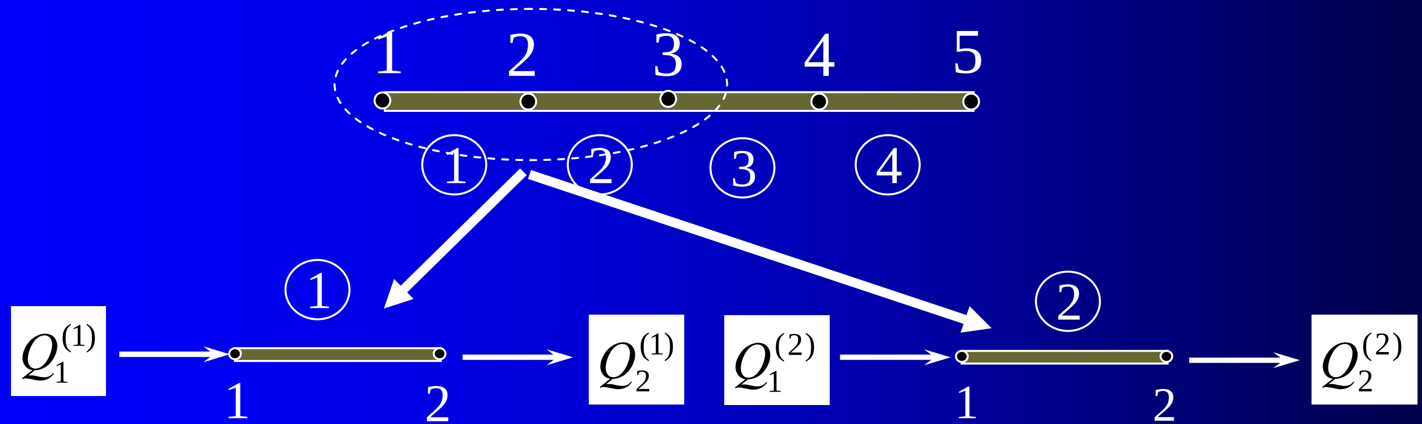


NUMERICAL EXAMPLE – 2 (continued)

(3) The boundary conditions are

$$\boxed{U_1 = 0, \quad U_5 = 0}$$

The equilibrium conditions are



$$\boxed{Q_2^{(1)} + Q_1^{(2)} = 0,\ Q_2^{(2)} + Q_1^{(3)} = 0,\ Q_2^{(3)} + Q_1^{(3)} = 0,\ Q_2^{(4)} + Q_1^{(4)} = 0}$$

NUMERICAL EXAMPLE – 2 (continued)

(4) The assembled equations are

$$\frac{1}{24}\begin{bmatrix} 94 & -97 & 0 & 0 & 0 \\ -97 & 188 & -97 & 0 & 0 \\ 0 & -97 & 188 & -97 & 0 \\ 0 & 0 & -97 & 188 & -97 \\ 0 & 0 & 0 & -97 & 94 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = -\begin{Bmatrix} 0.001302 \\ 0.018229 \\ 0.065104 \\ 0.143230 \\ 0.105470 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix} \begin{matrix} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \\ \end{matrix}$$

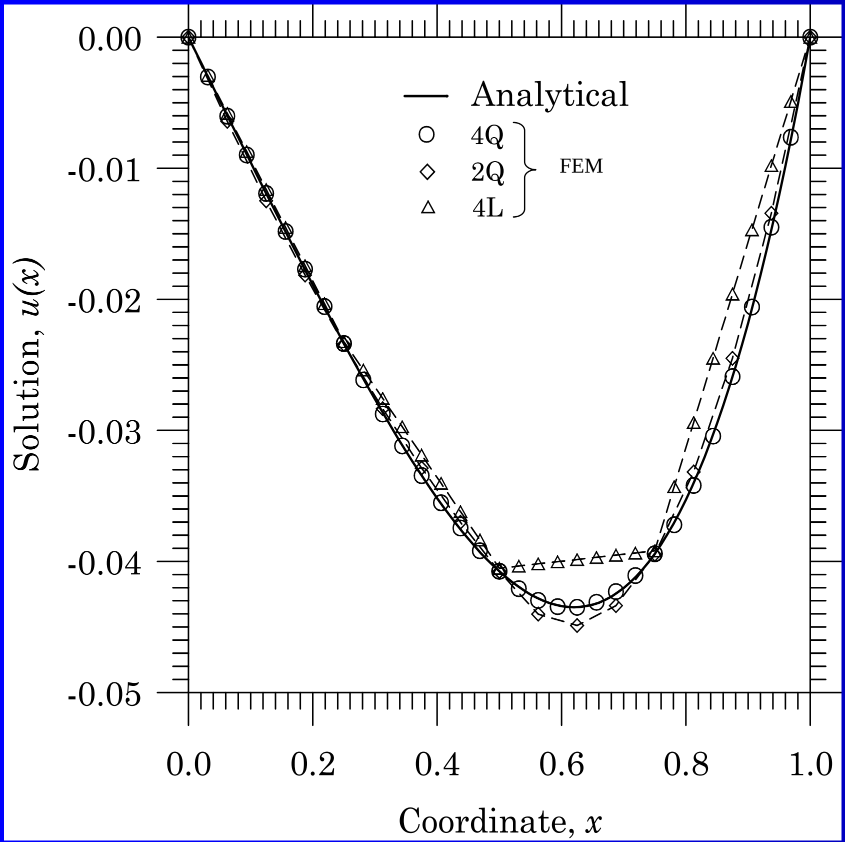
(5) The condensed equations for the unknown *U*'s and *Q*'s are

$$\begin{bmatrix} 7.8333 & -4.0417 & 0 \\ -4.0417 & 7.8333 & -4.0417 \\ 0 & -4.0417 & 7.8333 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = -\begin{Bmatrix} 0.01823 \\ 0.06510 \\ 0.14323 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1^1 \\ Q_2^4 \end{Bmatrix} = \begin{bmatrix} 3.9167 & -4.0417 & 0 \\ 0 & -4.0417 & 3.9167 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} + \begin{Bmatrix} 0.001302 \\ 0.10547 \end{Bmatrix} = \begin{Bmatrix} 0.09520 \\ 0.26386 \end{Bmatrix}$$

NUMERICAL EXAMPLE – 2 (continued)

Plot of the solution



Plot of the derivative of the solution