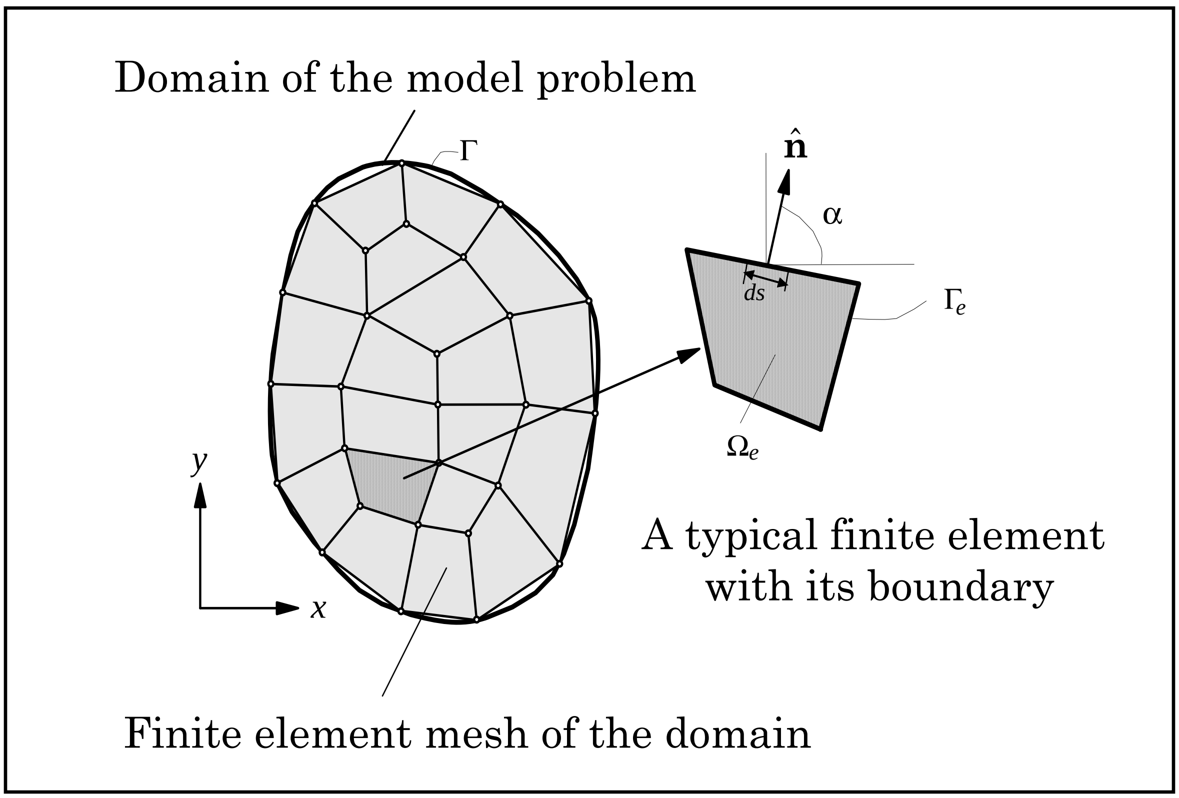
FINITE ELEMENT FORMULATION OF POISSON'S EQUATION IN TWO DIMENSIONS

Poisson's Equation is given by ($a_{12} = a_{21} = a_{00} = 0$ in the model equation of the book)

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)= f(x,y) \text{ in } \mathbf{\Omega}$$

where

a_{11}, a_{22}, and f are known functions of x and y.



STEP-BY-STEP FORMULATION OF THE PROBLEM

1. Model Differential Equation

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a_{22}\frac{\partial u}{\partial y}\right)=f(x,y) \tag{1}$$

2. Weak Form

$$0=\int_{\Omega^e}\left(a_{11}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+a_{22}\frac{\partial w}{\partial y}\frac{\partial u}{\partial y}-wf\right)dxdy-\oint_{\Gamma^e}wq_n ds \tag{2a}$$

$$q_n=a_{11}\frac{\partial u}{\partial x}n_x+a_{22}\frac{\partial u}{\partial y}n_y \tag{2b}$$

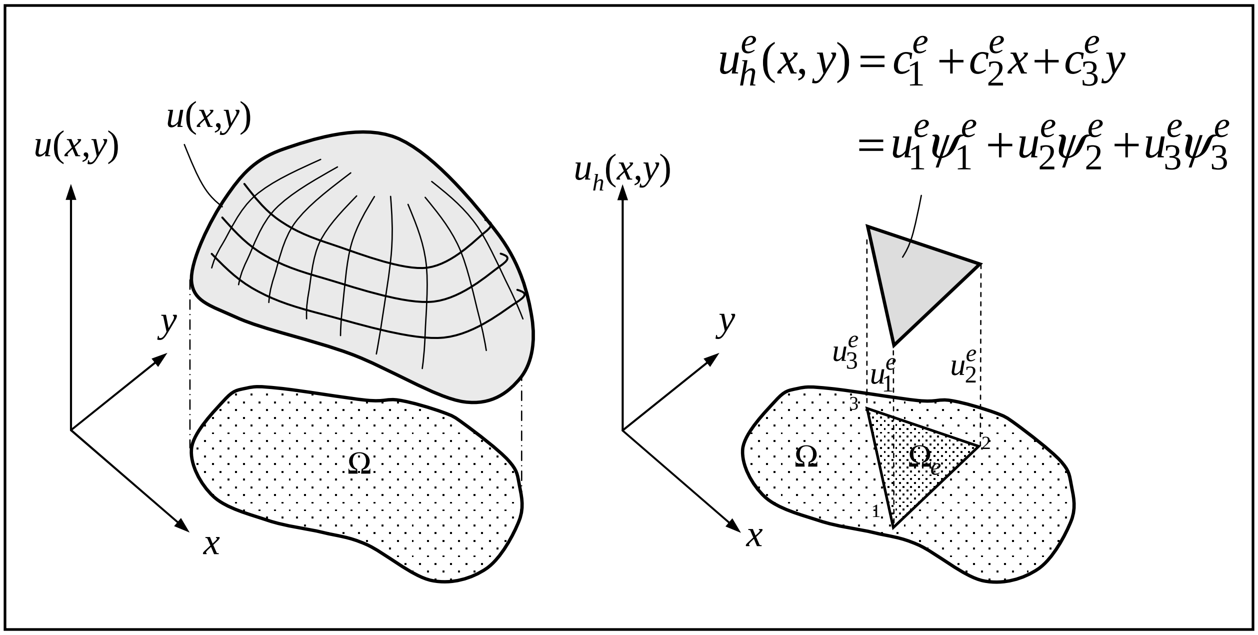
3. Finite Element Model

$$u(x,y)\approx\sum_{j=1}^{n}u_j\psi_j^e(x,y) \tag{3}$$

$$[K^e]\{u^e\}=\{f^e\}+\{Q^e\} \tag{4}$$

$$\begin{aligned}
K_{ij}^e &= \int_{\Omega^e}\left(a_{11}\frac{\partial \psi_i^e}{\partial x}\frac{\partial \psi_j^e}{\partial x}+a_{22}\frac{\partial \psi_i^e}{\partial y}\frac{\partial \psi_j^e}{\partial y}\right)dxdy\\
f_i^e &= \int_{\Omega^e}\psi_i^e f\ dxdy\\
Q_i^e &= \oint_{\Gamma^e}\psi_i^e q_n\ ds
\end{aligned} \tag{5}$$

Finite Element Approximation of the Solution



Finite Element Approximation of the Solution:

$$u_h^e(x,y) = \sum_{j=1}^{n} u_j^e\, \psi_j^e(x,y)$$

where n denotes the number of nodes in the element.

$$n = \begin{cases} 3, & \text{linear triangular element} \\ 4, & \text{linear rectantangular element} \\ 6, & \text{quadratic triangular element} \\ 8 \text{ or } 9 & \text{quadratic rectangular element} \end{cases}$$

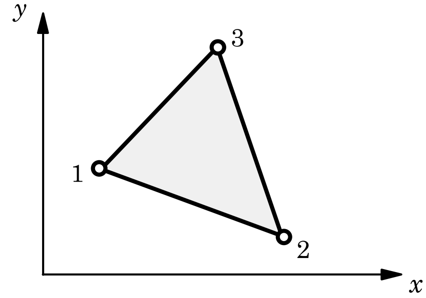
Linear Triangular Element

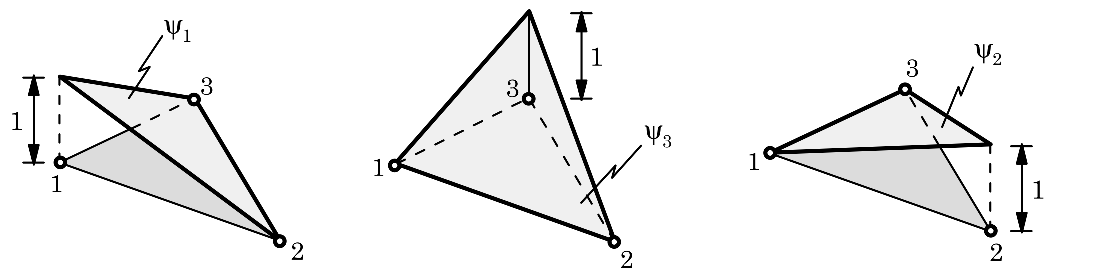
$$\psi_i(x,y) = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$\alpha_i = x_j y_k - x_k y_j,$$

$$\beta_i = y_j - y_k, \quad 2A = 2 \times \text{triangle area}$$

$$\gamma_i = -(x_j - x_k)$$





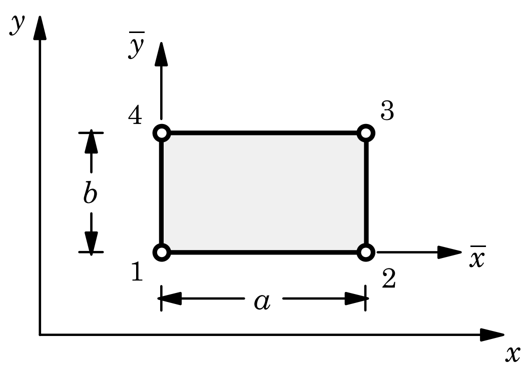
Linear Rectangular Element

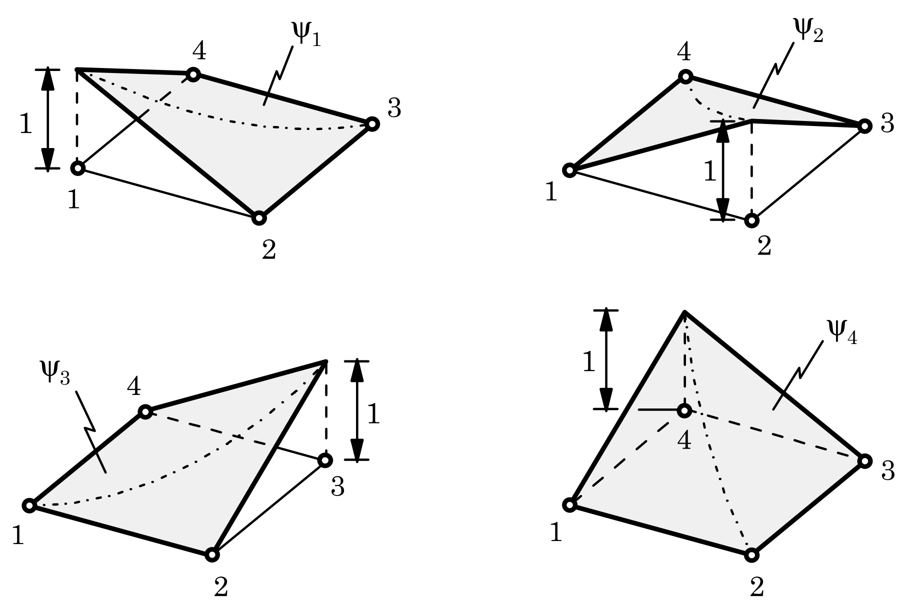
The general form of ψ_i is

$$\psi_i(x,y) = \alpha_i + \beta_i x + \gamma_i y + \lambda_i xy$$

$$\psi_1(\bar{x},\bar{y}) = \left(1-\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right), \quad \psi_2(\bar{x},\bar{y}) = \left(\frac{\bar{x}}{a}\right)\left(1-\frac{\bar{y}}{b}\right),$$

$$\psi_3(\bar{x},\bar{y}) = \left(\frac{\bar{x}}{a}\right)\left(\frac{\bar{y}}{b}\right), \quad \psi_4(\bar{x},\bar{y}) = \left(1-\frac{\bar{x}}{a}\right)\frac{\bar{y}}{b}$$





ELEMENT CALCULATIONS

Linear triangular element

For element-wise constant values of a_{11}^e, a_{22}^e, and f_0^e, we have

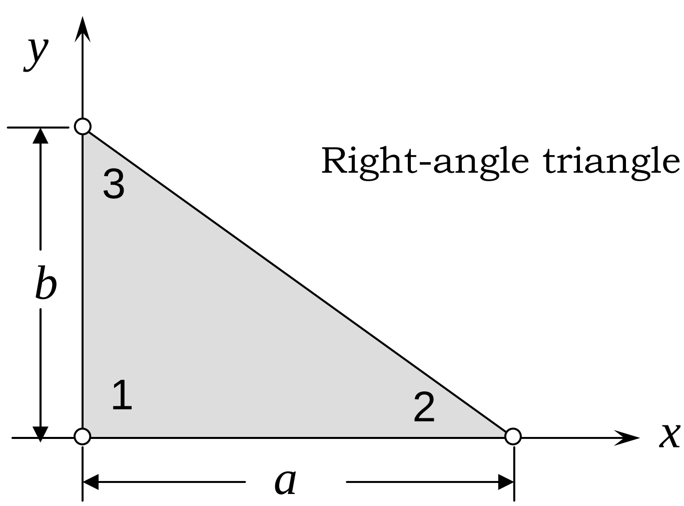
$$K_{ij}^e = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + a_{22} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy$$

$$= a_{11}^e \int_{\Omega^e} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} \, dxdy + a_{22}^e \int_{\Omega^e} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \, dxdy$$

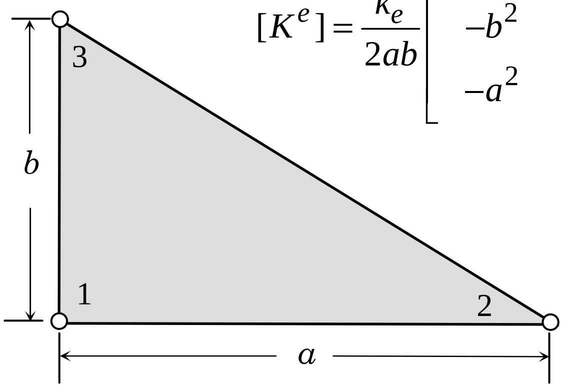
$$f_i^e = \int_{\Omega^e} \psi_i^e f \, dxdy = f_0^e \int_{\Omega^e} \psi_i^e \, dxdy = \frac{f_0^e A_T^e}{3}$$

$$[K^e] = \frac{a_{11}^e}{2a} \begin{bmatrix} b & -b & 0 \\ -b & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{a_{22}^e}{2b} \begin{bmatrix} a & 0 & -a \\ 0 & 0 & 0 \\ -a & 0 & a \end{bmatrix}$$

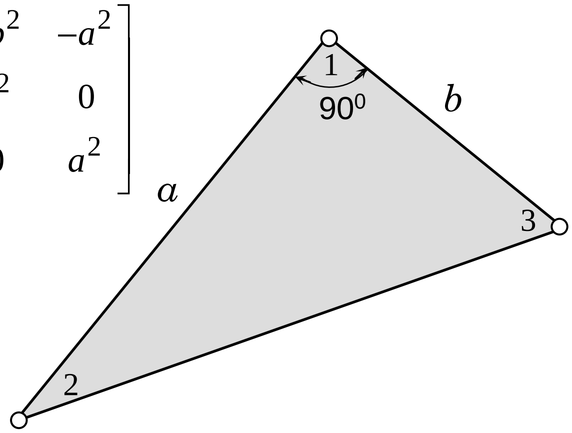
$$\{f^e\} = \frac{f_0^e A_T^e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad A_T^e = \text{ area of the triangle}$$



<u>Element Calculations</u> (continued)



$$[K^e] = \frac{k_e}{2ab}\begin{bmatrix} b^2+a^2 & -b^2 & -a^2 \\ -b^2 & b^2 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix}$$



When node numbers are renumbered, the coefficient matrix associated with the new node numbers can be readily obtained as explained below. For example, change node numbers as shown.

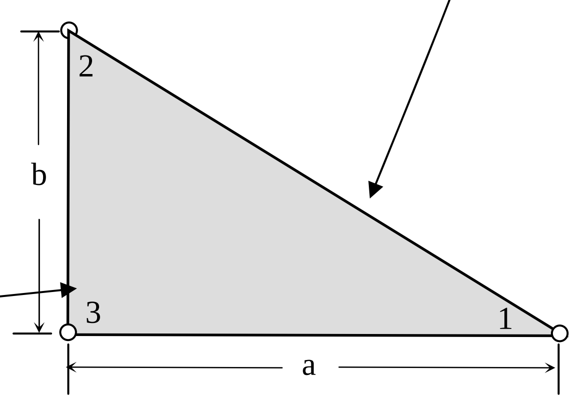
$$1 \rightarrow 3,\; 2 \rightarrow 1,\;\; 3 \rightarrow 2$$

$$[K^e] = \frac{k_e}{2ab}\begin{bmatrix} b^2+a^2 & -b^2 & -a^2 \\ -b^2 & b^2 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix} \text{Col 1} \leftrightarrow \text{Col 3:}\; \frac{k_e}{2ab}\begin{bmatrix} -a^2 & -b^2 & b^2+a^2 \\ 0 & b^2 & -b^2 \\ a^2 & 0 & -a^2 \end{bmatrix}$$

$$\text{Row 1} \leftrightarrow \text{Row 3:}\; \frac{k_e}{2ab}\begin{bmatrix} a^2 & 0 & -a^2 \\ 0 & b^2 & -b^2 \\ -a^2 & -b^2 & a^2+b^2 \end{bmatrix};\text{Col 2} \leftrightarrow \text{Col 1:}\; \frac{k_e}{2ab}\begin{bmatrix} 0 & a^2 & -a^2 \\ b^2 & 0 & -b^2 \\ -b^2 & -a^2 & a^2+b^2 \end{bmatrix}$$

$$\text{Row 2} \leftrightarrow \text{Row 1:}\; [K^e] = \frac{k_e}{2ab}\begin{bmatrix} b^2 & 0 & -b^2 \\ 0 & a^2 & -a^2 \\ -b^2 & -a^2 & a^2+b^2 \end{bmatrix}$$

This coefficient matrix corresponds to this numbering in a right-angled triangular element.



ELEMENT CALCULATIONS

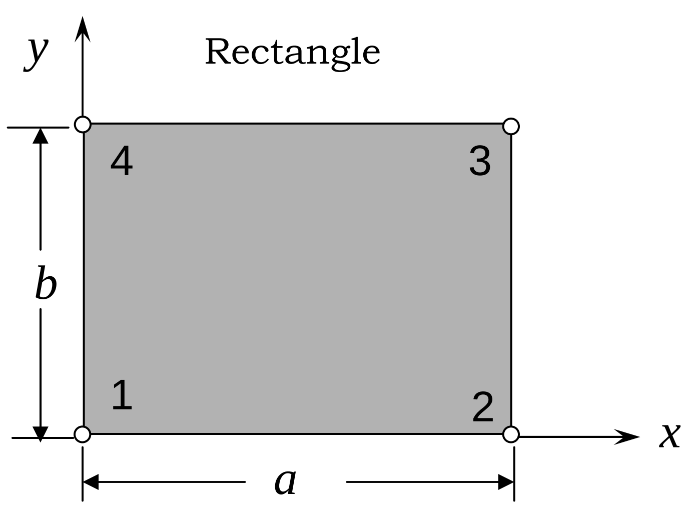
Linear rectangular element

For element-wise constant values of a_{11}^e, a_{22}^e, and f_0^e, we have

$$K_{ij}^e = \int_{\Omega^e} \left(a_{11} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} + a_{22} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \right) dxdy$$
$$= a_{11}^e \int_{\Omega^e} \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} \; dxdy + a_{22}^e \int_{\Omega^e} \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} \; dxdy$$
$$f_i^e = \int_{\Omega^e} \psi_i^e f \; dxdy = f_0^e \int_{\Omega^e} \psi_i^e \; dxdy = \frac{f_0^e A_R^e}{4}$$

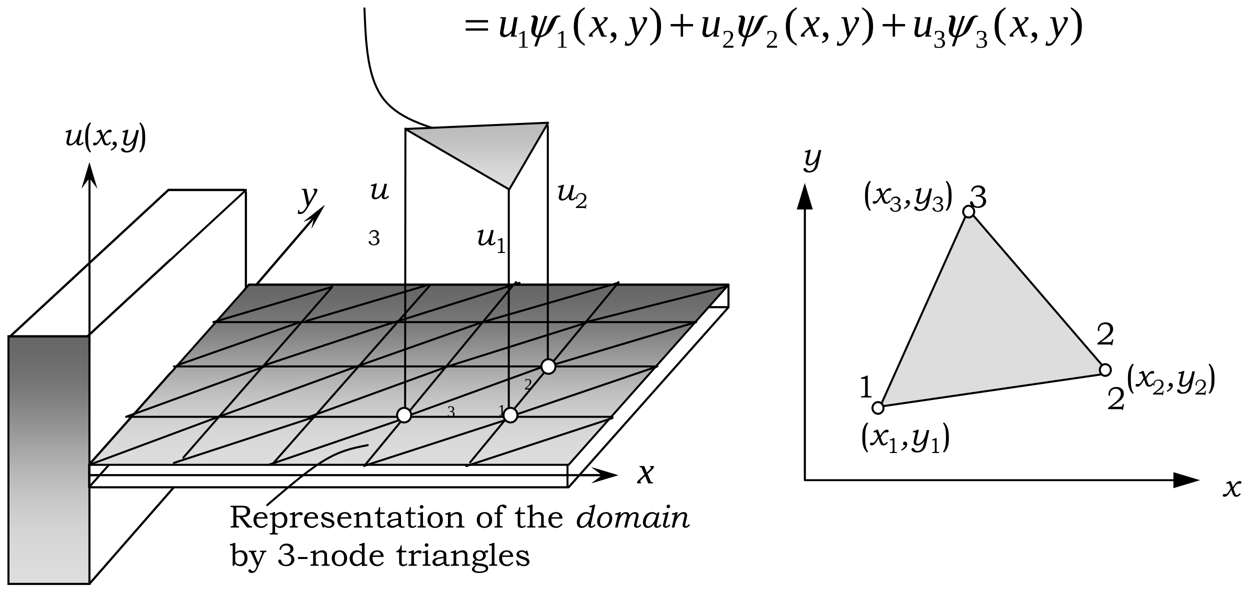
$$[K^e] = \frac{a_{11}}{6a} \begin{bmatrix} 2b & -2b & -b & b \\ -2b & 2b & b & -b \\ -b & b & 2b & -2b \\ b & -b & -2b & 2b \end{bmatrix} + \frac{a_{22}}{6b} \begin{bmatrix} 2a & -2a & -a & a \\ -2a & 2a & a & -a \\ -a & a & 2a & -2a \\ a & -a & -2a & 2a \end{bmatrix}$$

$$\{f^e\} = \frac{f_0^e A_R^e}{4} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}, \quad A_R^e = \text{ area of the rectangle}$$



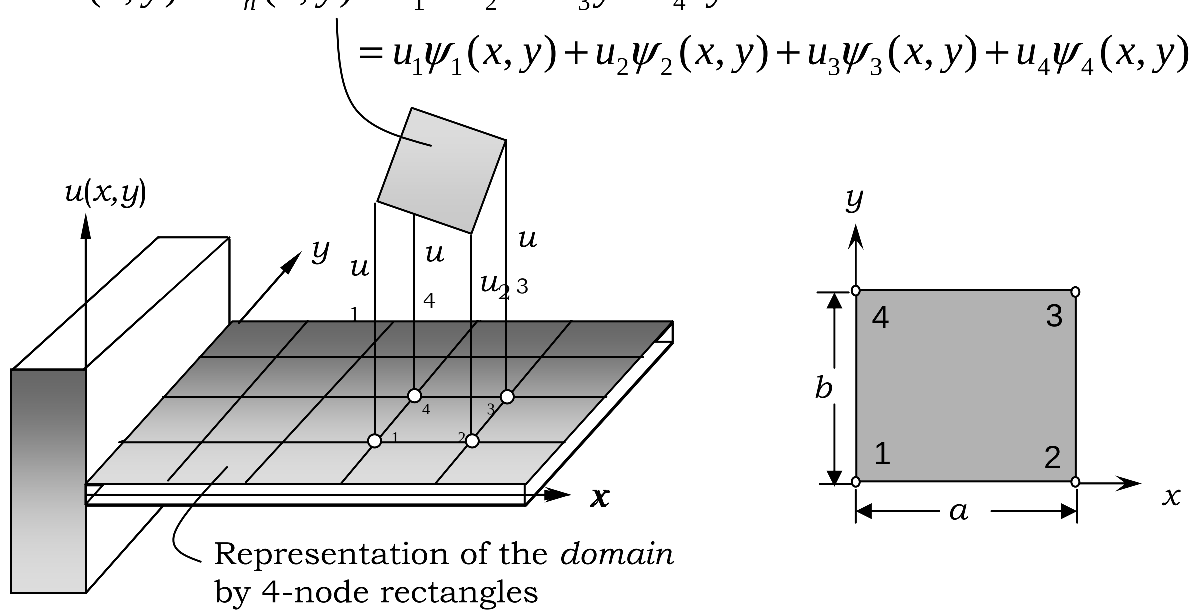
Representation of the *solution* by linear polynomials

$$u(x,y) \approx u_h^e(x,y) = c_1 + c_2 x + c_3 y$$
$$= u_1\psi_1(x,y) + u_2\psi_2(x,y) + u_3\psi_3(x,y)$$



Representation of the *solution* by bilinear polynomials

$$u(x,y) \approx u_h^e(x,y) = c_1 + c_2 x + c_3 y + c_4 xy$$
$$= u_1\psi_1(x,y) + u_2\psi_2(x,y) + u_3\psi_3(x,y) + u_4\psi_4(x,y)$$



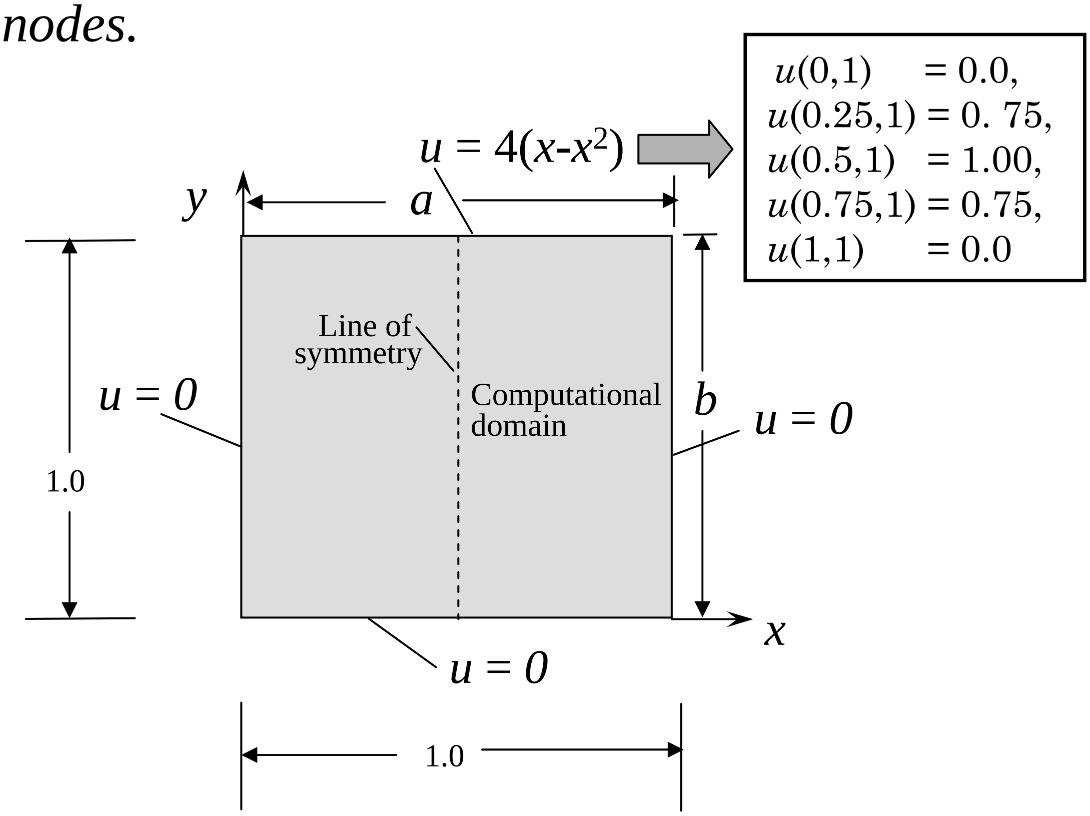
An Example

Problem 8.18

Given the following differential equation

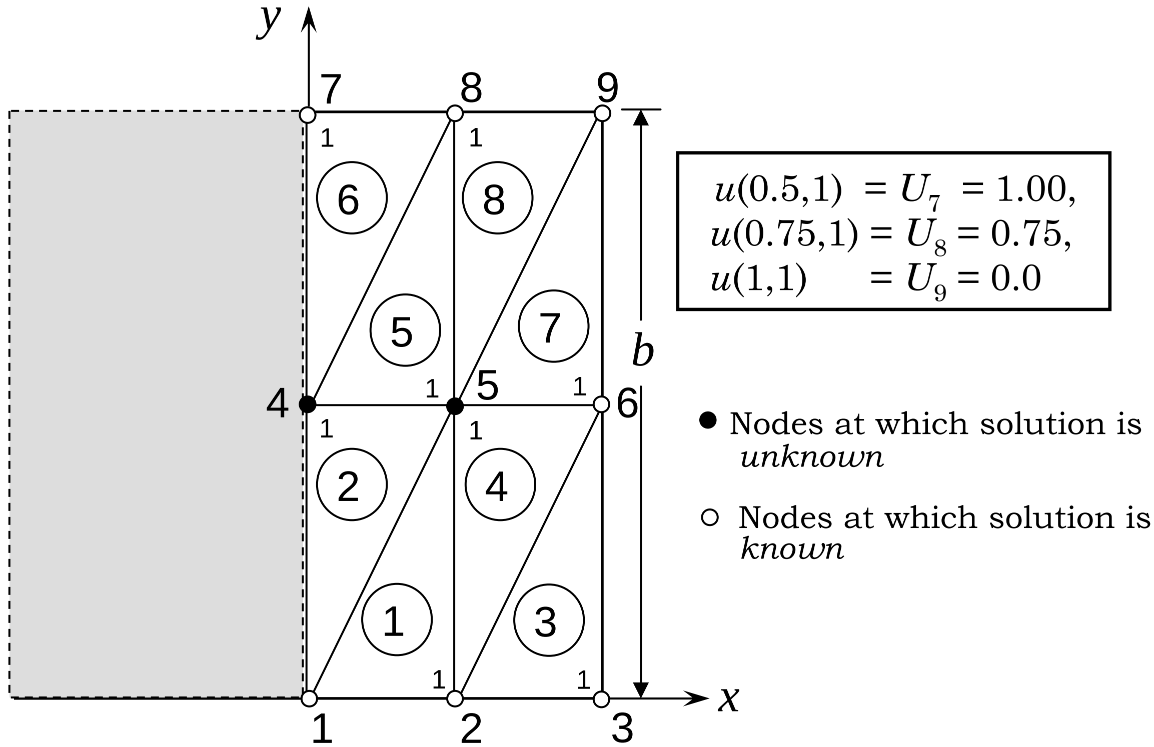
$$-k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f_0$$

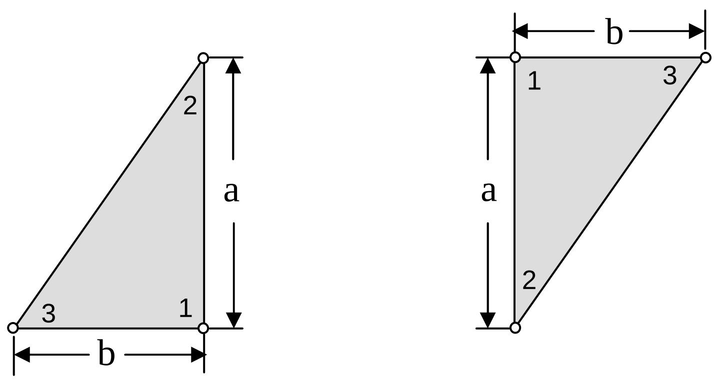
on the square domain and boundary conditions shown in the figure below, *determine the values of the unknown u at the nodes.*



$$a^e_{11}=a^e_{22}=k\ ;\ \ f^e_0=f_0=0$$

An Example (continued)





For $a = 0.5$ and $b = 0.25$, we have $(a/b = 2$ and $b/a = 0.5)$

$$[K^e] = \frac{k}{2}\begin{bmatrix} \frac{b}{a}+\frac{a}{b} & -\frac{b}{a} & -\frac{a}{b} \\ -\frac{b}{a} & \frac{b}{a} & 0 \\ -\frac{a}{b} & 0 & \frac{a}{b} \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 2.5 & -0.5 & -2.0 \\ -0.5 & 0.5 & 0.0 \\ -2.0 & 0.0 & 2.0 \end{bmatrix}$$

$$\{f^e\} = \frac{f_0 ab}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{0.125 f_0}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

<u>An Example</u> (continued)

- There are 9 global equations for the problem at hand.
- Since U_4 and U_5 are the only unknowns of the problem, it is sufficient to write global equations 4 and 5. We have

$$\begin{aligned} K_{41}U_1 + K_{42}U_2 + K_{43}U_3 + K_{44}U_4 + K_{45}U_5 + +K_{46}U_6 \\ + K_{47}U_7 + K_{48}U_8 + K_{49}U_9 = F_4 \\ K_{51}U_1 + K_{52}U_2 + K_{53}U_3 + K_{54}U_4 + K_{55}U_5 + +K_{56}U_6 \\ + K_{57}U_7 + K_{58}U_8 + K_{59}U_9 = F_5 \end{aligned}$$

- The boundary conditions on the primary variables are

$$U_1 = U_2 = U_3 = U_6 = U_9 = 0$$

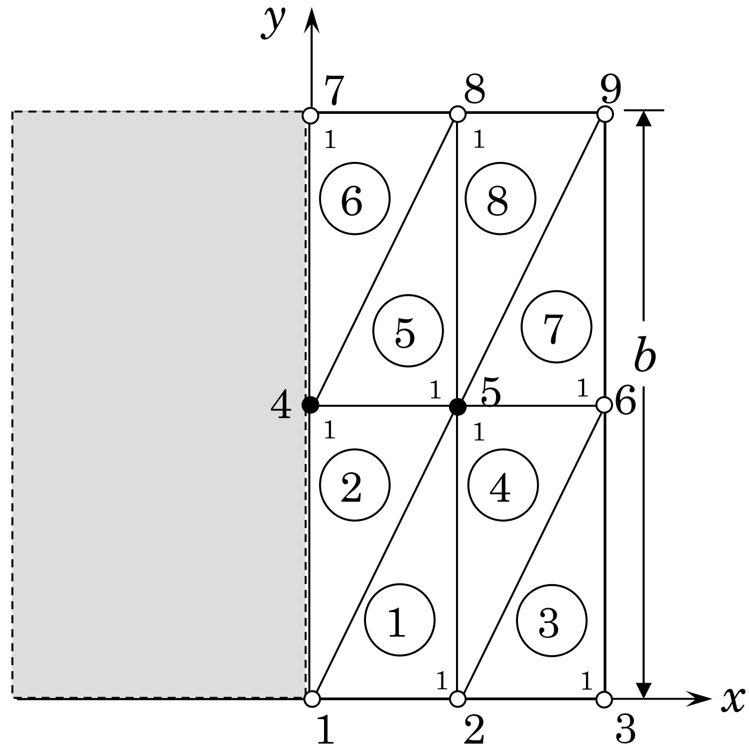
Hence, we do not have to compute their coefficients. We have

$$\begin{aligned} K_{44}U_4 + K_{45}U_5 &= F_4 - (K_{47}U_7 + K_{48}U_8) \\ K_{54}U_4 + K_{55}U_5 &= F_5 - (K_{57}U_7 + K_{58}U_8) \end{aligned}$$

- The global coefficients K_{IJ} and F_I can be expressed in terms of the element coefficients K_{ij}^e, f_i^e, and Q_i^e. For ALL elements ($e = 1, 2, \cdots, 8$), we have

$$[K^e] = \frac{k}{2}\begin{bmatrix} 2.5 & -0.5 & -2.0 \\ -0.5 & 0.5 & 0.0 \\ -2.0 & 0.0 & 2.0 \end{bmatrix}, \quad \{f^e\} = \frac{0.125 f_0}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

An Example (continued)



$$K_{44} = K_{11}^2 + K_{33}^5 + K_{22}^6 = \frac{k}{2}\,(2.5 + 2.0 + 0.5) = 2.5k$$

$$K_{45} = K_{13}^2 + K_{31}^5 = \frac{k}{2}\,(-2.0 - 2.0) = -2.0k$$

$$K_{47} = K_{21}^6 = \frac{k}{2}\,(-0.5) = -0.25k$$

$$K_{48} = K_{23}^6 + K_{32}^5 = \frac{k}{2}\,(0.0 + 0.0) = 0.0$$

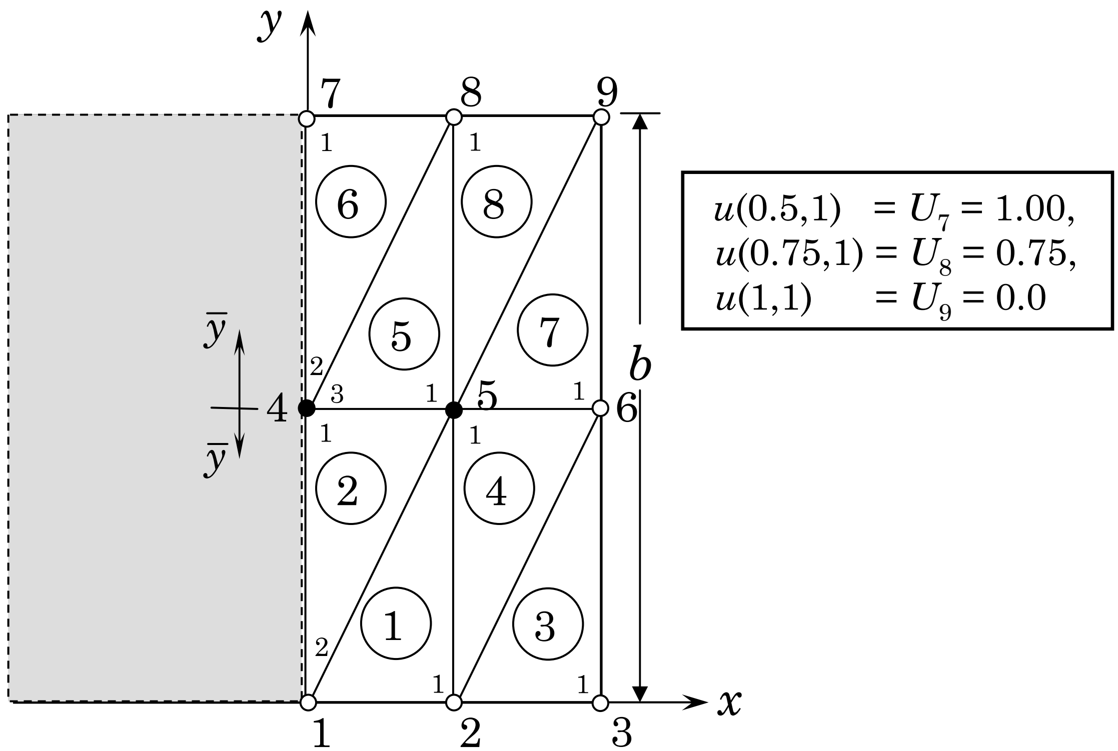
$$K_{55} = K_{22}^1 + K_{33}^2 + K_{11}^4 + K_{11}^5 + K_{33}^7 + K_{22}^8$$

$$= \frac{k}{2}\,(0.5 + 2.0 + 2.5 + 2.5 + 2.0 + 0.5) = 5.0k$$

$$K_{57} = 0$$

$$K_{58} = K_{12}^5 + K_{21}^8 = \frac{k}{2}\,(-0.5 - 0.5) = -0.5k$$

<u>An Example</u> (continued)



$$F_4 = f_1^2 + f_3^5 + f_2^6 + Q_1^2 + Q_3^5 + Q_2^6$$
$$= \frac{0.125 f_0}{6}(1.0 + 1.0 + 1.0) + \int_0^{0.5} q_n \psi_1^{(2)}\, d\bar{y} + \int_0^{0.5} q_n \psi_2^{(6)}\, d\bar{y}$$
$$= 0.0625 f_0 + 0$$
$$F_5 = f_2^1 + f_3^2 + f_1^4 + f_1^5 + f_3^7 + f_2^8 + Q_2^1 + Q_3^2 + Q_1^4 + Q_1^5 + Q_3^7 + Q_2^8$$
$$= \frac{0.125 f_0}{6}(1.0 + 1.0 + 1.0 + 1.0 + 1.0 + 1.0) + 0 = 0.125 f_0$$

Final ***condensed equations*** for the unknown *primary nodal variables*

$$k\begin{bmatrix} 2.5 & -2.0 \\ -2.0 & 5.0 \end{bmatrix}\begin{Bmatrix} U_4 \\ U_5 \end{Bmatrix} = f_0\begin{Bmatrix} 0.25 \\ 0.5 \end{Bmatrix} - k\begin{Bmatrix} (-0.25)\times 1.0 \\ (-0.5)\times 0.75 \end{Bmatrix}$$

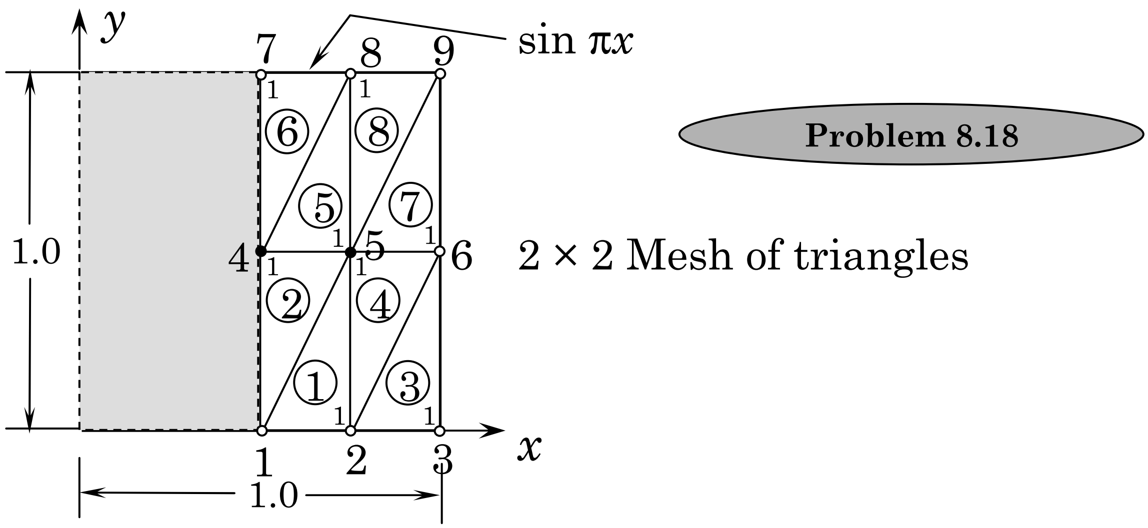


Table: Comparison of FEM solution with the exact solution (meshes of linear triangular elements).

x	y	2×2	4×4	8×8	Exact
0.5000	0.125	-	-	0.0355	0.0349
0.5000	0.250	-	0.0797	0.0764	0.0752
0.5000	0.375	-	-	0.1291	0.1273
0.5000	0.500	0.2303	0.2080	0.2015	0.1993
0.5000	0.625	-	-	0.3050	0.3024
0.5000	0.750	-	0.4630	0.4554	0.4527
0.5000	0.875	-	-	0.6758	0.6737

Table: Comparison of FEM solution with the exact solution (meshes of linear rectangular elements).

x	y	2×2	4×4	8×8	Exact
0.5000	0.125	-	-	0.0343	0.0349
0.5000	0.250	-	0.0703	0.0740	0.0752
0.5000	0.375	-	-	0.1255	0.1273
0.5000	0.500	0.1520	0.1895	0.1969	0.1993
0.5000	0.625	-	-	0.2996	0.3024
0.5000	0.750	-	0.4410	0.4499	0.4527
0.5000	0.875	-	-	0.6716	0.6737