Two-Dimensional Frame Elements

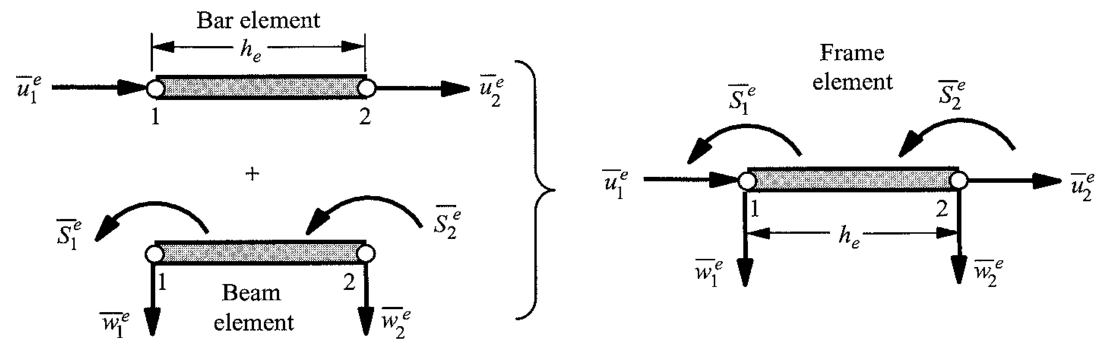

Figure 5.4.1 Superposition of bar and beam element to obtain a frame element [degrees of freedom are referred to the element coordinate system $(\bar{x}, \bar{y}, \bar{z})$].

A superposition of the bar element of Section 4.6 with the EBE of Section 5.2 or the Timoshenko beam element (RIE, CIE, or IIE) of Section 5.3 gives a frame element with three primary degrees of freedom (u, w, S) per node (note that the transverse displacement v of Section 4.6 is now denoted by w to be consistent with Sections 5.2 and 5.3). When the axial stiffness EA and bending stiffness EI are elementwise constant, the superposition of the linear bar element with the IIE gives the following element equations (see Fig. 5.4.1):

$$[\bar{K}]^e\{\bar{\Delta}\}^e = \{\bar{F}\}^e \tag{5.4.1a}$$

or, in explicit form,

$$\frac{2EI}{\mu_0 h^3}\begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3h & 0 & -6 & -3h \\ 0 & -3h & h^2(1.5+6\Lambda) & 0 & 3h & h^2(1.5-6\Lambda) \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3h & 0 & 6 & 3h \\ 0 & -3h & h^2(1.5-6\Lambda) & 0 & 3h & h^2(1.5+6\Lambda) \end{bmatrix}^e \begin{Bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{S}_1 \\ \bar{u}_2 \\ \bar{w}_2 \\ \bar{S}_2 \end{Bmatrix}^e = \begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \\ \bar{F}_4 \\ \bar{F}_5 \\ \bar{F}_6 \end{Bmatrix}^e \tag{5.4.1b}$$

where

$$\begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \bar{F}_3 \\ \bar{F}_4 \\ \bar{F}_5 \\ \bar{F}_6 \end{Bmatrix}^e = \begin{Bmatrix} \bar{f}_1 \\ \bar{q}_1 \\ \bar{q}_2 \\ \bar{f}_2 \\ \bar{q}_3 \\ \bar{q}_4 \end{Bmatrix}^e + \begin{Bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \bar{Q}_3 \\ \bar{Q}_4 \\ \bar{Q}_5 \\ \bar{Q}_6 \end{Bmatrix}^e \tag{5.4.2a}$$

$$\bar{f}_i = \int_0^{h_e} f^e(\bar{x})\psi_i^e(\bar{x})\, d\bar{x} \quad (i = 1, 2), \quad \bar{q}_i = \int_0^{h_e} q^e(\bar{x})\phi_i^e(\bar{x})\, d\bar{x} \quad (i = 1, 2, 3, 4) \tag{5.4.2b}$$

and

$$\mu = \frac{A\mu_0 h^2}{2I}, \quad \Lambda = \frac{EI}{GAK_s h^2}, \quad \mu_0 = 12\Lambda \tag{5.4.3}$$

In Eq. (5.4.2*b*), f^e denotes the distributed axial force, q^e the distributed transverse force, ψ_i^e the linear interpolation functions, and ϕ_i^e the Hermite cubic interpolation functions. In the following paragraphs, we develop transformation relations to express the element equations (5.4.1*b*)—valid in the element coordinate system ($\bar{x}$, $\bar{y}$, $\bar{z}$)—to the global coordinate system (x, y, z).

The local coordinates ($\bar{x}_e$, $\bar{y}_e$, $\bar{z}_e$) of a typical element Ω_e are related to the global coordinates (x, y, z) by [cf. Eq. (4.6.2)]

$$\begin{Bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^e \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{5.4.4}$$

where the angle α_e is measured clockwise from the global x-axis to the element $\bar{x}_e$-axis. Note that the y and $\bar{y}_e$ coordinates are parallel to each other, and they are *out of* the plane of the paper (see Fig. 5.4.2). The same transformation relations hold for displacements (u, w) along the global coordinates (x, z) and displacements ($\bar{u}$,$\bar{w}$) in the local coordinates

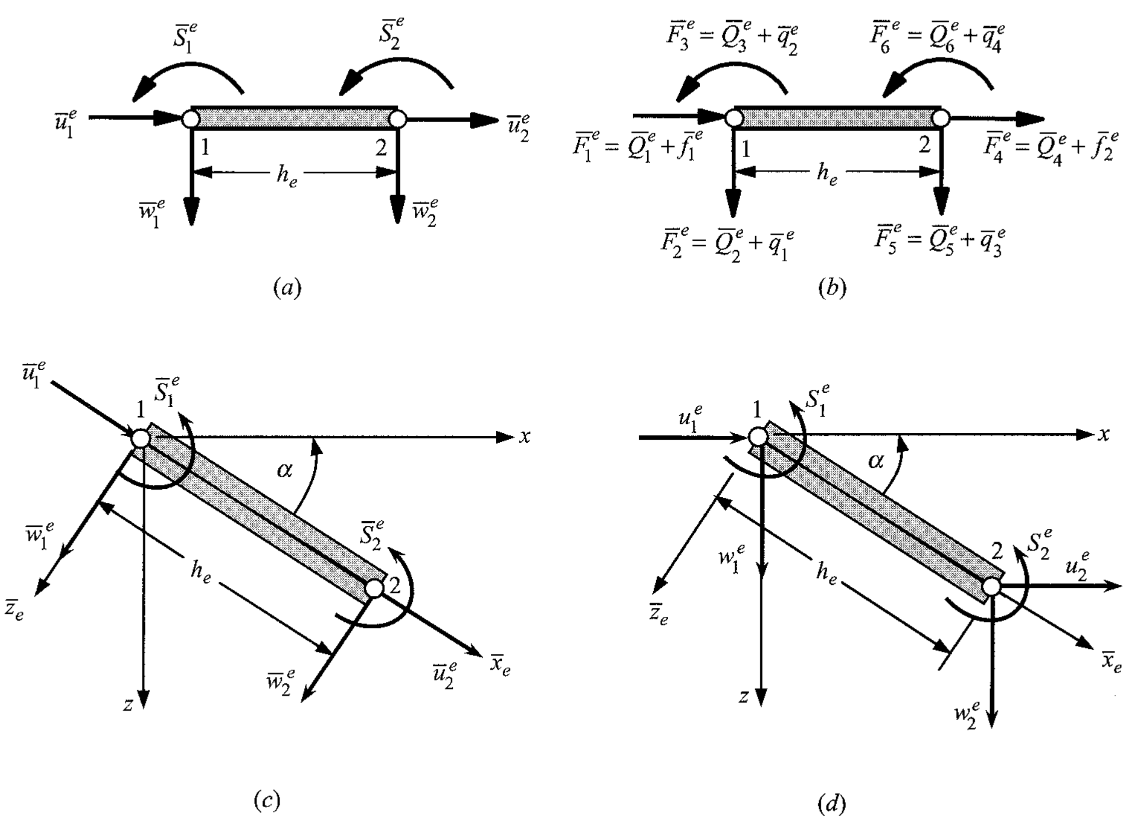

Figure 5.4.2 (*a*) Generalized displacements. (*b*) Generalized forces. (*c*) Generalized displacements in the element coordinates. (*d*) Generalized displacements in the global coordinates.

$(\bar{x}, \bar{z})$. Note that there is no displacement in the direction of the coordinate y (i.e., $v = 0$). However, there is a rotation about the y-axis, and it remains the same in both coordinate systems because $y = \bar{y}$. Note that rotation θ is equal to $-dw/dx$ in Euler–Bernoulli beam theory and it is equal to Ψ in Timoshenko beam theory. Hence, the relationship between (u, w, θ) and $(\bar{u}, \bar{w}, \bar{\theta})$ can be written as

$$\begin{Bmatrix} \bar{u} \\ \bar{w} \\ \bar{\theta} \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^e \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix}^e \tag{5.4.5}$$

Therefore, the three nodal degrees of freedom $(\bar{u}_i^e, \bar{w}_i^e, \bar{S}_i^e)$ at the ith node $(i = 1, 2)$ in the $(\bar{x}, \bar{y}, \bar{z})$ system are related to the three degrees of freedom (u_i^e, w_i^e, S_i^e) in the (x, y, z) system by

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{w}_1 \\ \bar{S}_1 \\ \bar{u}_2 \\ \bar{w}_2 \\ \bar{S}_2 \end{Bmatrix}^e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & & & \\ -\sin\alpha & \cos\alpha & 0 & & \mathbf{0} & \\ 0 & 0 & 1 & & & \\ & & & \cos\alpha & \sin\alpha & 0 \\ & \mathbf{0} & & -\sin\alpha & \cos\alpha & 0 \\ & & & 0 & 0 & 1 \end{bmatrix}^e \begin{Bmatrix} u_1 \\ w_1 \\ S_1 \\ u_2 \\ w_2 \\ S_2 \end{Bmatrix}^e \tag{5.4.6a}$$

or

$$\{\bar{\Delta}^e\} = [T^e]\{\Delta^e\} \tag{5.4.6b}$$

Analogously, the element force vectors in the local and global coordinate systems are related according to

$$\{\bar{F}\}^e = [T]^e\{F\}^e \tag{5.4.7}$$

Returning to Eq. (5.4.1a), we substitute the transformation equations (5.4.6b) and (5.4.7) into (5.4.1a) and obtain

$$[\bar{K}]^e[T]^e\{\Delta\}^e = [T]^e\{F\}^e$$

Premultiplying both sides with $[T]^{-1} = [T]^{\mathrm{T}}$, we obtain

$$[T]^{\mathrm{T}}[\bar{K}]^e[T]^e\{\Delta\}^e = \{F\}^e \quad \text{or} \quad [K]^e\{\Delta\}^e = \{F\}^e \tag{5.4.8}$$

where

$$[K^e] = [T]^{\mathrm{T}}[\bar{K}]^e[T]^e, \quad \{F\}^e = [T]^{\mathrm{T}}\{\bar{F}\}^e \tag{5.4.9}$$

Thus, if we know the element matrices $[\bar{K}]^e$ and $\{\bar{F}\}^e$ of an element Ω_e in the local coordinate system $(\bar{x}, \bar{y}, \bar{z})$, the element matrices in the global coordinate system are obtained by (5.4.9).

Using $[\bar{K}]^e$ and $\{\bar{F}\}^e$ from Eq. (5.4.1b) in (5.4.9) and carrying out the indicated matrix multiplications, we arrive at the following element stiffness matrix $[K^e]$ referred to the global coordinates:

$$[K]^e = \frac{2EI}{\mu_0 h^3} \begin{bmatrix} \mu\cos^2\alpha + 6\sin^2\alpha & (\mu-6)\cos\alpha\sin\alpha & 3h\sin\alpha & -(\mu\cos^2\alpha + 6\sin^2\alpha) & -(\mu-6)\cos\alpha\sin\alpha & 3h\sin\alpha \\ (\mu-6)\cos\alpha\sin\alpha & \mu\sin^2\alpha + 6\cos^2\alpha & -3h\cos\alpha & -(\mu-6)\sin\alpha\cos\alpha & -(\mu\sin^2\alpha + 6\cos^2\alpha) & -3h\cos\alpha \\ 3h\sin\alpha & -3h\cos\alpha & h^2(1.5+6\Lambda) & -3h\sin\alpha & 3h\cos\alpha & h^2(1.5-6\Lambda) \\ -(\mu\cos^2\alpha + 6\sin^2\alpha) & -(\mu-6)\sin\alpha\cos\alpha & -3h\sin\alpha & (\mu\cos^2\alpha + 6\sin^2\alpha) & (\mu-6)\cos\alpha\sin\alpha & -3h\sin\alpha \\ -(\mu-6)\cos\alpha\sin\alpha & -(\mu\sin^2\alpha + 6\cos^2\alpha) & 3h\cos\alpha & (\mu-6)\cos\alpha\sin\alpha & \mu\sin^2\alpha + 6\cos^2\alpha & 3h\cos\alpha \\ 3h\sin\alpha & -3h\cos\alpha & h^2(1.5-6\Lambda) & -3h\sin\alpha & 3h\cos\alpha & h^2(1.5+6\Lambda) \end{bmatrix} \tag{5.4.10a}$$

$$\{F\}^e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}^e = \begin{Bmatrix} \bar{F}_1\cos\alpha - \bar{F}_2\sin\alpha \\ \bar{F}_1\sin\alpha + \bar{F}_2\cos\alpha \\ \bar{F}_3 \\ \bar{F}_4\cos\alpha - \bar{F}_5\sin\alpha \\ \bar{F}_4\sin\alpha + \bar{F}_5\cos\alpha \\ \bar{F}_6 \end{Bmatrix}^e \tag{5.4.10b}$$

which is the element force vector referred to the global coordinates.