Exercise 4.24 Solution

Multiply differential equation by test function w and integrate over element $\Rightarrow$

$$\int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right]dx + \int_{x_a}^{x_b} w\left[\frac{d}{dx}\left(EA\alpha T\right)\right]dx - \int_{x_a}^{x_b} wf\,dx = 0$$

Integrate by parts on first integral term to get the weak form $\Rightarrow$

$$\int_{x_a}^{x_b} EA\frac{du}{dx}\frac{dw}{dx}dx + \int_{x_a}^{x_b} w\left[\frac{d}{dx}\left(EA\alpha T\right) - f\right]dx - wEA\frac{du}{dx}\bigg|_{x_a}^{x_b} = 0$$

Note temperature term stays with distributed loading term f

Using Ritz - Galerkin scheme, weak form reduces to element equation $[K]\{u\} = \{F\}$ with matrices

$$K_{ij}^e = \int_{x_a}^{x_b} EA\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx\ ,\ F_i^e = \int_{x_a}^{x_b}[f - \frac{d}{dx}(EA\alpha T)\psi_i]dx + Q_i^e$$

Using linear approximating functions : $\psi_1 = \dfrac{x_b - x}{h}\ ,\psi_2 = \dfrac{x - x_a}{h}$ and $A(x) = 6 - \dfrac{x}{10}\ ,\ f = 0 \Rightarrow$

$$[K] = \frac{E}{h}\left(6 - \frac{2x_a + h}{20}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ \{F\} = \frac{E\alpha Th}{20}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

For the bar problem using three linear finite elements with $U_1 = 0$, the condensed equations become

$$\frac{E}{h}\begin{bmatrix} \left(12 - \frac{h}{5}\right) & -\left(6 - \frac{3h}{20}\right) & 0 \\ \bullet & \left(12 - \frac{2h}{5}\right) & -\left(6 - \frac{5h}{20}\right) \\ \bullet & \bullet & \left(6 - \frac{5h}{20}\right) \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \frac{E\alpha Th}{20}\begin{Bmatrix} 2 \\ 2 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

Boundary and Matching Conditions : $Q_2^3 = P = 400lb\ ,\ Q_2^1 + Q_1^2 = Q_2^2 + Q_1^3 = 0$

Using $E = 30x10^6\,lb/in^2\ ,\ \alpha = 12x10^{-6}\,/in\,^oF\ ,\ h = 10in\ ,\ T = 60^oF \Rightarrow$

$$\begin{bmatrix} 30 & -13.5 & 0 \\ \bullet & 24 & -10.5 \\ \bullet & \bullet & 10.5 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0.0216 \\ 0.0216 \\ 0.01121 \end{Bmatrix} \Rightarrow \text{Solution using MATLAB} \Rightarrow \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0.0033 \\ 0.0057 \\ 0.0068 \end{Bmatrix}$$

```
% Exercise 4.24 – Solution to Global System
clc
K=[30,-13.5,0;-13.5,24,-10.5;0,-10.5,10.5]
F=[0.0216;0.0216;0.0112]
U=K\F
```