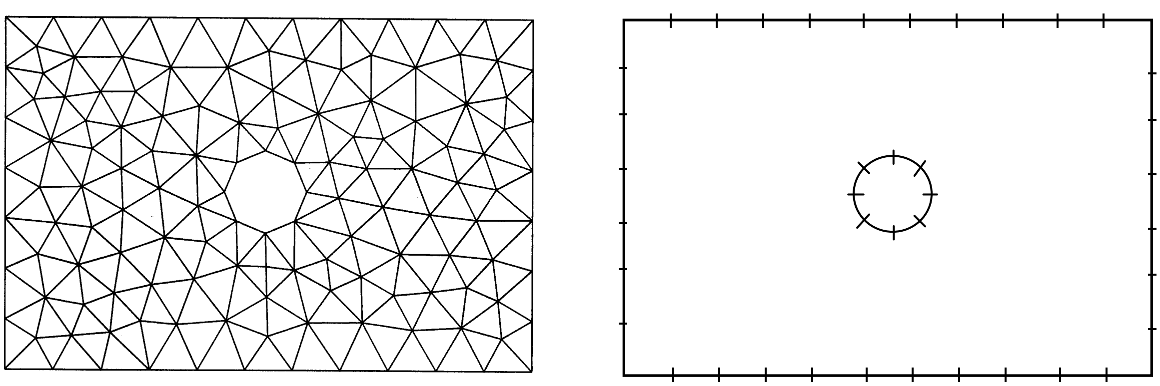
Boundary Element Method for Elasticity Problems

Another general numerical method has recently emerged that provides good computational abilities and has some particular advantages when compared to FEM. The technique known as the *boundary element method* (BEM) has been widely used by computational mechanics investigators leading to the development of many private and commercial codes. Similar to the finite element method, BEM can analyze many different problems in engineering science including those in thermal sciences and fluid mechanics. Although the method is not limited to elastic stress analysis, our brief presentation will only discuss this particular case. Many texts have been written that provide additional details on this subject, see for example Banerjee and Butterfield (1981) and Brebbia and Dominguez (1992).

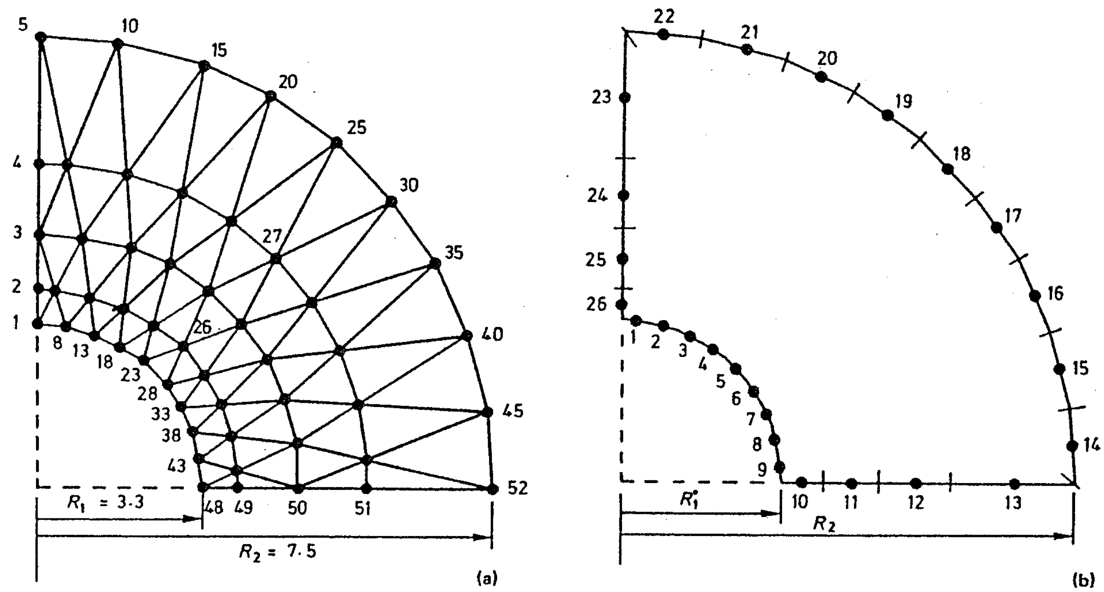
The formulation of BEM is based on an integral statement of elasticity, and this can be cast into a relation involving unknowns only over the boundary of the domain under study. This originally lead to the boundary integral equation method (BIE), and early work in the field was reported by Rizzo (1967) and Cruse (1969). Subsequent research realized that finite element methods could be used to solve the boundary integral equation by dividing the boundary into elements over which the solution is approximated using appropriate interpolation functions. This process generates an algebraic system of equations to solve for the unknown nodal values that approximate the boundary solution. A procedure to calculate the solution at interior domain points can also be determined from the original boundary integral equation. This scheme also allows variation in element size, shape and approximating scheme to suit the application, thus providing similar advantages as FEM.

By discretizing only the boundary of the domain, BEM has particular advantages over FEM. The first issue is that the resulting BEM equation system is generally much smaller than that generated by finite elements. It has been pointed out in the literature, that boundary discretization is somewhat easier to interface with CAD computer codes that create the original problem geometry. Two-dimensional comparisons of equivalent FEM and BEM meshes for a rectangular plate with a central circular hole, hollow cylinder and gear tooth problem and shown in Figures 1 and 2. It is apparent that a significant reduction in the number of elements (by a factor of five) is realized in the BEM mesh.

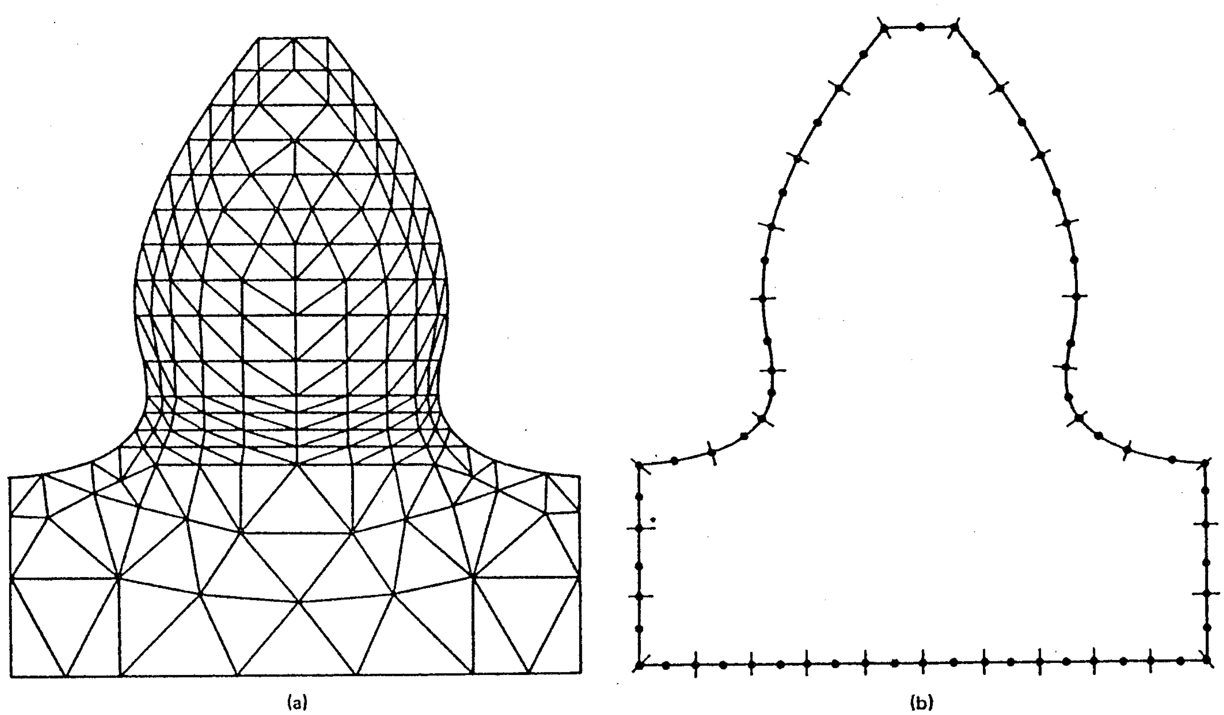


(FEM Discretization: 228 Elements) (BEM Discretization: 44 Elements)

Figure 1. Comparison of Typical FEM and BEM Meshes



(a) Finite element mesh versus (b) boundary element mesh



(a) Discretization of gear tooth into 291 finite elements. (b) Discretization of gear tooth into 33 boundary elements.

Figure 2. Comparisons of FEM and BEM meshes.

It should be pointed out however, that the BEM scheme will not automatically compute the solution at interior points, and thus additional computational effort is required to find such information. Some studies have indicated that BEM more accurately determines stress concentration effects. Problems of infinite extent (e.g. full-space or half-space domains) create some difficulty in developing appropriate FEM meshes, whereas particular BEM schemes can automatically handle the infinite nature of the problem and only require limited boundary meshing. There exists several additional advantages and disadvantages related to each method; however we will not pursue further comparison and debate. For linear elasticity, both methods offer considerable utility to numerically solve very complex problems. We now proceed with a brief development of the boundary element method for two-dimensional elasticity problems.

An integral statement of the elasticity field equations (sometimes called *Somigliana's identity*) can be developed. This is normally done using the *reciprocal theorem*, whereby one elastic state is selected as the fundamental solution while the other state is chosen as the desired solution field. For a region V with boundary S, this leads to the following result

$$cu_j(\boldsymbol{\xi}) = \int_S [T_i(\boldsymbol{x})G_{ij}(\boldsymbol{x},\boldsymbol{\xi}) - u_i T_{ikj}(\boldsymbol{x},\boldsymbol{\xi})n_k]dS + \int_V F_i G_{ij}(\boldsymbol{x},\boldsymbol{\xi})dV \tag{1}$$

where the coefficient c is given by

$$c = \begin{cases} 1, \ \xi \ in \ V \\ \frac{1}{2}, \xi \ on \ S \\ 0, \ \xi \ outside \ V \end{cases} \tag{2}$$

$G_{ij}(\boldsymbol{x},\xi)$ is the *displacement Green's function* that comes from the fundamental solution to the elasticity equations, and corresponds to the solution of the displacement field at point $\boldsymbol{x}$ produced by a unit concentrated body force $\boldsymbol{e}$ located at point ξ

$$u_i^G(\boldsymbol{x}) = G_{ij}(\boldsymbol{x};\boldsymbol{\xi})\boldsymbol{e}_j(\boldsymbol{\xi}) \tag{3}$$

The stresses associated with this state are specified by

$$\sigma_{ij}^G = T_{ijk}(\boldsymbol{x},\xi)\boldsymbol{e}_k = [\lambda G_{lk,l}\delta_{ij} + \mu(G_{ik,j} + G_{jk,i})]\boldsymbol{e}_k \tag{4}$$

and the tractions follow to be

$$T_i^G = \sigma_{ij}^G n_j = T_{ijk} n_j \boldsymbol{e}_k = p_{ik}\boldsymbol{e}_k \tag{5}$$

with $p_{ik} = T_{ijk}n_j$. Relation (1) gives the displacement of a given observational point ξ in terms of boundary and volume integrals. If point ξ is chosen to lie on boundary S then the expression will contain unknowns (displacements and tractions) only on the boundary. For this case (ξ on S), relation (2) indicates $c = ½$, but this is true only if the boundary has a continuous tangent (i.e., is smooth). Slight modifications are necessary for the case of non-smooth boundaries, see Brebbia and Dominguez (1992).

Restricting our attention to only the two-dimensional plane strain case, the Green's function becomes (Brebbia and Dominguez, 1992)

$$G_{ij} = \frac{1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{ij} + r_{,i}r_{,j}\right] \tag{6}$$

where $r = |\,\boldsymbol{x} - \boldsymbol{\xi}\,|$ is the distance between points $\boldsymbol{x}$ and $\boldsymbol{\xi}$. Relation (5) can be used to determine the traction p_{ij} associated with this specific Green's function giving the result

$$p_{ij} = T_{ikj}n_k = -\frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)\left(\frac{\partial r}{\partial n}\delta_{ij} + r_{,j}n_i - r_{,i}n_j\right) + 2\frac{\partial r}{\partial n}r_{,i}r_{,j}\right] \tag{7}$$

It will be convenient to use matrix notation in the subsequent formulation and thus define

$$\boldsymbol{G} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},\ \boldsymbol{p} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$
$$\boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\ \boldsymbol{T} = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix},\ \boldsymbol{F} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{8}$$

The boundary integral equation (1) can then be expressed in two-dimensional form as

$$c^i\boldsymbol{u^i} = \int_\Gamma [\boldsymbol{GT} - \boldsymbol{pu}]dS + \int_R \boldsymbol{GF}\,dV \tag{9}$$

It is noted that by allowing point ξ to be on the boundary, this relation will contain unknown displacements or tractions only over Γ. We now wish to apply numerical finite element concepts to solve (9) by discretizing the boundary Γ and region R into subdomains over which the solution will be approximated. Only the simplest case will be presented here in which the approximating scheme assumes piecewise constant values for the unknowns.

Referring to Figure 3, a typical boundary Γ is discretized into N elements. The unknown boundary displacements and tractions are assumed to be constant over each element and equal to the value at each mid-node. Subdivision of the interior into cells would also be required in order to compute integration of the body force term over R. However, such interior integrals can be re-formulated in terms of boundary integrals, thereby maintaining efficiency of the basic boundary techniques. This re-formulation will not be discussed here, and we will now formally drop body force contributions from further consideration. Using this discretization scheme, relation (9) can be written as

$$c^i\boldsymbol{u^i} + \sum_{j=1}^{N}\left(\int_{\Gamma_j}\boldsymbol{p}\,ds\right)\boldsymbol{u^j} = \sum_{j=1}^{N}\left(\int_{\Gamma_j}\boldsymbol{G}\,ds\right)\boldsymbol{T}^{\,j} \tag{10}$$

where index i corresponds to a particular node where the Green's function concentrated force is applied, and index j is related to each of the boundary elements including the case $j = i$. Notice that for the choice of constant approximation over the element, there is no formal interpolation function required, and nodal values are simply brought outside of the element integrations.

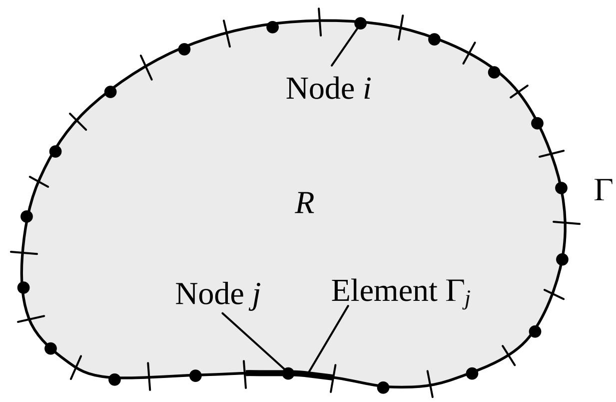


Figure 3. Boundary Discretization Using Elements With Constant Approximation

Reviewing the previous expressions (6) and (7), the integral terms $\int_{\Gamma_j} \boldsymbol{G}\, ds$ and $\int_{\Gamma_j} \boldsymbol{p}\, ds$ relate node i to node j and are sometimes referred to as influence functions. Each of these terms generate 2x2 matrices which can be defined by

$$\hat{A}^{ij} = \int_{\Gamma_j} \boldsymbol{p}\, ds$$
$$B^{ij} = \int_{\Gamma_j} \boldsymbol{G}\, ds \qquad (11)$$

For the constant element case, some of the integrations in (11) can be carried out analytically, while other cases use numerical integration commonly employing Gauss quadrature. It should be noted that the $i = j$ case generates a singularity in the integration, and special methods are normally used to handle this problem.

Relation (10) can thus be written as

$$c^i \boldsymbol{u}^{\boldsymbol{i}} + \sum_{j=1}^{N} \hat{A}^{ij} \boldsymbol{u}^{\boldsymbol{j}} = \sum_{j=1}^{N} B^{ij} \boldsymbol{T}^{\boldsymbol{j}} \qquad (12)$$

and this result specifies the value of $\boldsymbol{u}$ at node i in terms of values of $\boldsymbol{u}$ and $\boldsymbol{T}$ at all other nodes on the boundary. If the boundary is smooth, $c^i = ½$ at all nodes. By defining

$$A^{ij} = \begin{cases} \hat{A}^{ij}, & i \neq j \\ \hat{A}^{ij} + c^i, & i = j \end{cases} \qquad (13)$$

equation (12) can be written in compact form as

$$\sum_{j=1}^{N} A^{ij} \boldsymbol{u}^{\boldsymbol{j}} = \sum_{j=1}^{N} B^{ij} \boldsymbol{T}^{\boldsymbol{j}} \qquad (14)$$

or in matrix form

$$[\boldsymbol{A}]\{\boldsymbol{u}\} = [\boldsymbol{B}]\{\boldsymbol{T}\} \qquad (15)$$

Boundary conditions from elasticity theory normally specify either the displacements or tractions or a mixed combination of the two variables over boundary Γ. Using these specified values in (14) or (15), reduces the number of unknowns and allows the system to be rearranged. Placing all unknowns on the left-hand side of the system equation and moving all known variables to the right, generates a final system that can always be expressed in the form

$$[\boldsymbol{C}]\{\boldsymbol{X}\} = \{\boldsymbol{D}\} \tag{16}$$

where all unknown boundary displacements and tractions are located in the column matrix $\{\boldsymbol{X}\}$ and all known boundary data have been multiplied by the appropriated influence function and moved into $\{\boldsymbol{D}\}$. Relation (16) represents a system of linear algebraic equations that can be solved for the desired unknown boundary information. This BEM system is generally much smaller than that from a corresponding FEM model. However, unlike the FEM system, the $[\boldsymbol{C}]$ matrix from (16) is not in general symmetric, thereby requiring more computational effort to solve for nodal unknowns. Using modern computing systems, usually this added computational effort is not a significant factor in the solution of linear elasticity problems.

Once this solution is complete, all boundary data is known and the solution at any desired interior point can be calculated reusing the basic governing boundary integral equation. For example, at an interior point relation (9) with no body forces will take the form

$$\boldsymbol{u}^{\boldsymbol{i}} = \int_\Gamma \boldsymbol{G}\boldsymbol{T}\,dS - \int_\Gamma \boldsymbol{p}\boldsymbol{u}\,dS \tag{17}$$

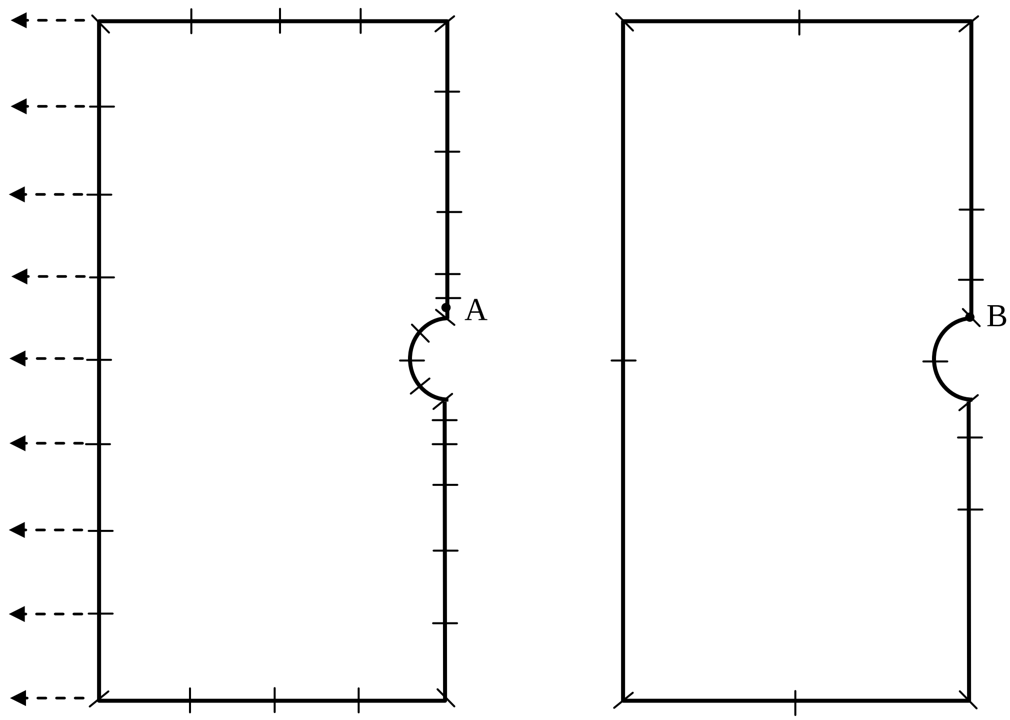
Following our previous constant element approximation, this expression can be discretized as

$$\begin{aligned} \boldsymbol{u}^{\boldsymbol{i}} &= \sum_{j=1}^{N}\left(\int_{\Gamma_j}\boldsymbol{G}\,ds\right)\boldsymbol{T}^{\boldsymbol{j}} - \sum_{j=1}^{N}\left(\int_{\Gamma_j}\boldsymbol{p}\,ds\right)\boldsymbol{u}^{\boldsymbol{j}} \\ &= \sum_{j=1}^{N} B^{ij}\,\boldsymbol{T}^{\boldsymbol{j}} - \sum_{j=1}^{N}\hat{A}^{ij}\,\boldsymbol{u}^{\boldsymbol{j}} \end{aligned} \tag{18}$$

and the interior displacement can then be determined using standard computational evaluation of the influence functions $\hat{A}^{ij}$ and B^{ij}. Internal values of strain and stress can also be computed using (17) in the strain-displacement relations and Hooke's law thereby generating expressions similar to relation (18), see Brebbia and Dominguez (1992).

Example BEM Solution of Circular Hole in a Plate Under Uniform Tension

Consider the problem of a plate under uniform tension containing a stress-free circular hole. The finite element solution was shown in a previous PowerPoint slide. A simple BEM FORTRAN code using constant and quadratic elements is provided in the text by Brebbia and Dominguez (1992) and this was used to develop the numerical solution. This simple BEM code does not have drawing or auto-meshing capabilities, and thus problem data was input by hand. The boundary element solution was generated using two different models that incorporated problem symmetry to analyze only half of the domain as shown in Figure 4. One model used 32 constant elements while the second case used 14 three-noded quadratic elements.



(32 Constant Element Mesh) (14 Quadratic Element Mesh)

Figure 4 BEM Solution of a Plate Under Uniform Tension Containing a Circular Hole

The constant element model is limited to having nodes located at the mid-point of each element (see Figure 4), and thus does not allow direct determination of the highest stress at the edge of the hole. For this case using the stress value at node A in the figure, the stress concentration factor was found to be $K \approx 2.75$. The quadratic element case uses three nodes per element including nodes located at the element boundaries. For this case node B was used to determine the highest stress and this gave a stress concentration factor of $K \approx 3.02$. The particular model had a width to diameter ratio of ten, and for such geometry results from Peterson (1974) would predict a stress concentration of about 3. As expected the BEM results using constant elements were not as good as the predictions using quadratic interpolation. Comparing this BEM analysis with a corresponding finite element model indicates that the quadratic boundary element results appear to give a more accurate estimate of the actual stress concentration using much fewer elements. However, this conclusion is based on the particular element models for each analysis, and using other element types and meshes could produce somewhat different results.

Example Hollow Cylinder Under Internal Pressure

Consider next the plane-strain problem of a thick-walled hollow cylinder under internal pressure as shown in Figure 5. The pressure is taken to be p = 100N/mm^2, while the internal and external radii are r_1 = 10mm and r_2 = 25mm, respectively. The elastic constants are E = 200kN/mm^2 and ν = 0.25. Using symmetry, only one quarter of the problem is analyzed employing 4, 10 and 15 quadratic elements. The Table presents the radial displacements of points A, B and C for each model. It is interesting to note that even the coarse mesh results are within 2% of the exact solution.

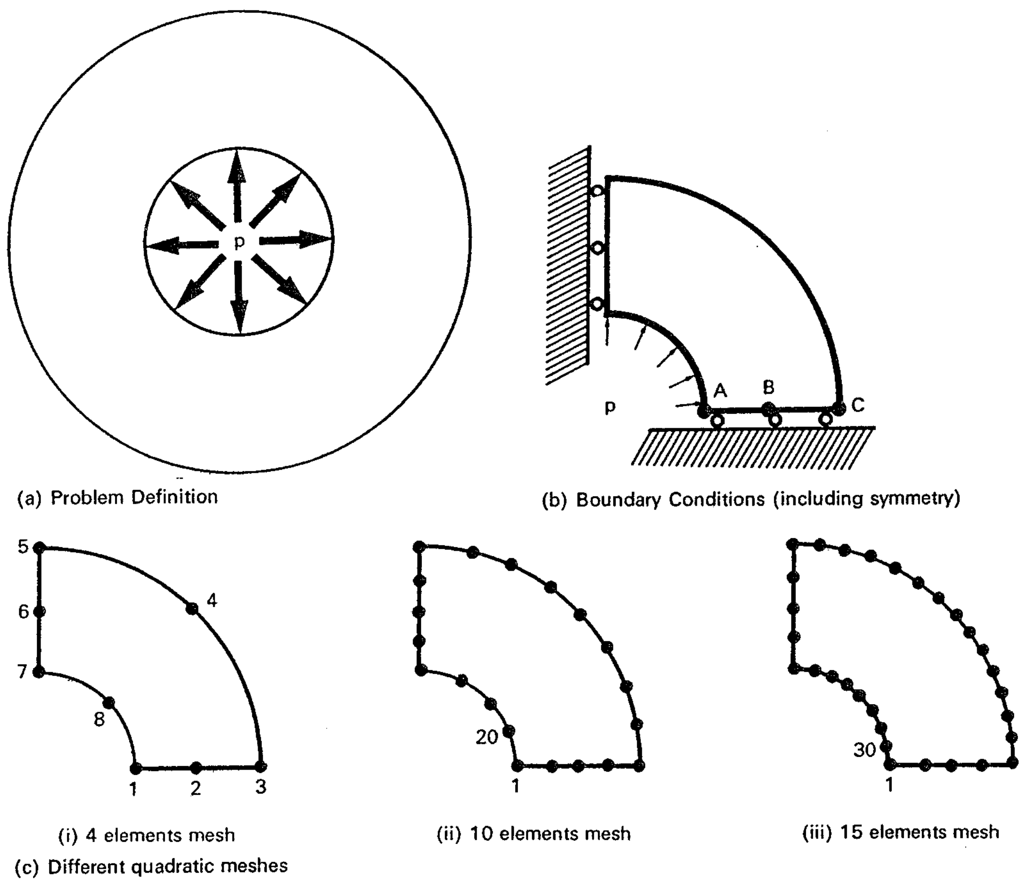
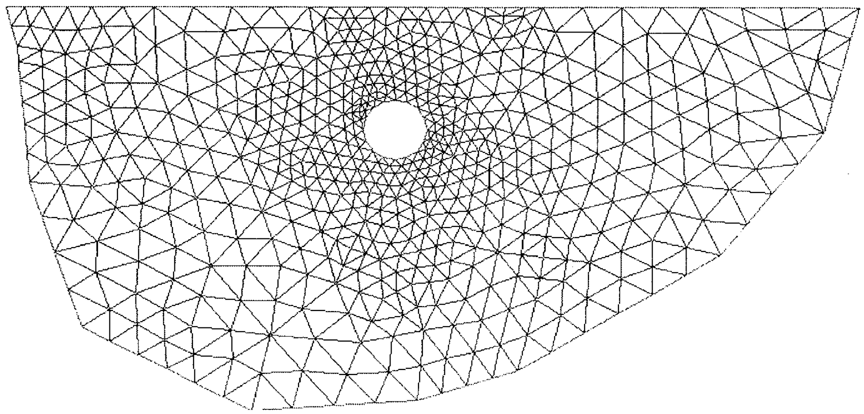


Figure 5. Hollow Cylinder Problem

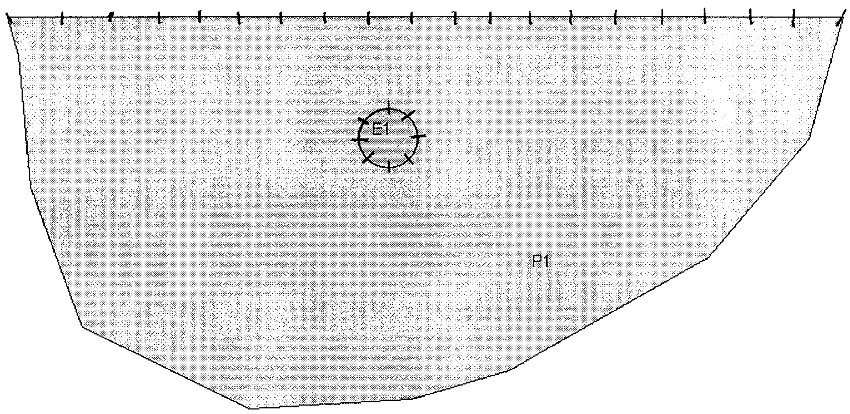
Table Radial Displacements for Hollow Cylinder under Internal Pressure (in 10^{-3} mm)

Node	Exact value	Discretization		
		4 elements	10 elements	15 elements
A	8.0325	7.8781	8.0246	8.0350
B	5.2912	5.1668	5.2845	5.2928
C	4.4526	4.3896	4.4520	4.4631

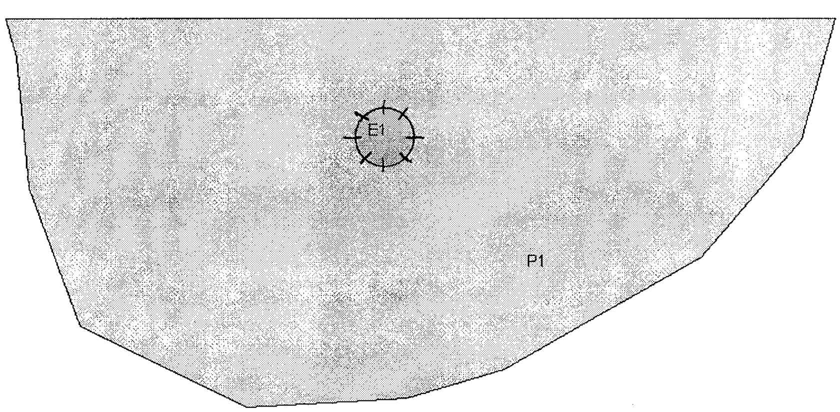
BEM Advantages Using Special Fundamental/Green's Functions



Finite element interior discretization resulting in numerous elements to model unbounded region



Boundary element discretization using infinite space fundamental solution resulting in having to discretize both hole and free surface



Boundary element discretization using semi-infinite space fundamental solution resulting in having to discretize only the interior hole

References

Banerjee, P.K. and Butterfield, R., *Boundary Element Methods in Engineering Science,* McGraw-Hill, New York, 1981.
Brebbia, C.A. and Dominguez, J., *Boundary Elements An Introductory Course,* WIT Press, Southampton, UK, 1992.
Cruse, T.A., Numerical Solutions in Three-Dimensional Elastostatics, Int. Jour. Solids Structures, Vol. 5, pp. 1259-1274, 1969.
Peterson, R.E., *Stress Concentration Factors,* John Wiley, New York, 1974.
Rizzo, R.J., An Integral Equation Approach to Boundary Value Problems of Classical Elastostatics, Quart. Appl. Math., Vol. 25, pp. 83- , 1967.