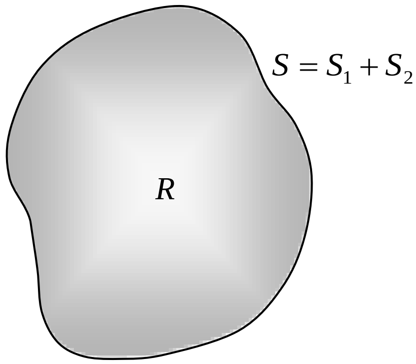
Boundary Integral Equation for Laplace's Equation

We wish to solve the following boundary value problem



$$\nabla^2 u = 0 \in R$$
$$u = \bar{u} \in S_1 \text{ and } q = \frac{\partial u}{\partial n} = \bar{q} \in S_2 \tag{1}$$

This can be accomplished in a somewhat different scheme by using some standard methods of applied mathematics. Using the divergence theorem, we can write

$$\iiint_R \nabla \cdot \boldsymbol{w}\, dV = \iint_S \boldsymbol{w} \cdot \boldsymbol{n}\, dS \tag{2}$$

Letting $\boldsymbol{w} = f\nabla g \Rightarrow \nabla \cdot \boldsymbol{w} = f\nabla^2 g + \nabla f \cdot \nabla g$ and $\boldsymbol{w} \cdot \boldsymbol{n} = f(\boldsymbol{n} \cdot \nabla g)$ and using these results in relation (2) gives

$$\iiint_R (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV = \iint_S f \frac{\partial g}{\partial n} dS \tag{3}$$

which is commonly called *Green's first formula or identity*. Next, interchanging *f* and *g* in equation (3) and subtracting this result from (3) results in

$$\iiint_R (f\nabla^2 g - g\nabla^2 f)\, dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dS \tag{4}$$

Relation (4) is sometimes known as *Green's second formula or identity*.

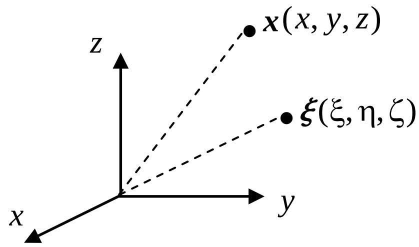
Now getting back to our original problem (1), let *f* in our previous relations be the solution to Laplace's equation $\nabla^2 f = 0$. Also let *g* be the *fundamental solution* (sometimes called the *Green's function*) associated with Laplace's equation

$$\nabla^2 g + \delta = 0 \tag{5}$$

where $\delta = \delta(\boldsymbol{x} - \boldsymbol{\xi})$ is the *Dirac delta function* with the property

$$\int F(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{\xi})d\boldsymbol{x} = F(\boldsymbol{\xi})$$

where $\boldsymbol{x}$ and $\boldsymbol{\xi}$ represent different points in the domain space.



Using these results for f and g in relation (4) gives

$$-f(\boldsymbol{\xi}) = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dS \tag{6}$$

which is often referred to as *Green's third formula or identity*.

The Green's function for the Laplace equation is well known from previous applied mathematics studies, and the two and three-dimensional expressions are given by

$$g = \left\{ \begin{matrix} \frac{1}{2\pi}\ln(1/r) \; . \; . \; . \; 2-D \\ \frac{1}{4\pi r} \quad . \; . \; . \; 3-D \end{matrix} \right\} \; , \; r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \tag{7}$$

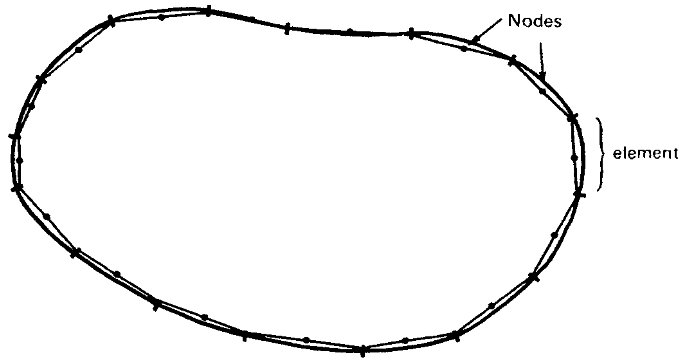
Letting $u = f$, $q = \dfrac{\partial u}{\partial n}$, $u^* = g$, and $q^* = \dfrac{\partial u^*}{\partial n}$, relation (6) gives

$$u(\xi) = \iint_S (u^* q - u q^*) dS \tag{8}$$

Since some of the boundary information for u and/or q will not be known, at present we cannot use equation (8) to solve the problem to determine the solution for u at any generic point. However, letting point $\boldsymbol{\xi}$ go to boundary yields the relation

$$\frac{1}{2}u(\xi) + \iint_S u q^* dS = \iint_S q u^* dS \tag{9}$$

where the factor of $\frac{1}{2}$ comes from the limiting case for a *smooth* boundary. Since equation (9) now contains only terms on the boundary, it is commonly called the *boundary integral equation (BIE) for Laplace's equation.* A general problem will have unknown values for u and/or q over some portions of the boundary. In order to solve for these unknowns, we can seek an approximate solution by using the standard procedures used in finite element analysis, and thus create a *boundary element method*. This is done by dividing the boundary into sub-sections (elements) and approximating the unknowns over each element thereby generating a system of equations in terms of nodal values (see figure below). Once all boundary unknowns are determined, equation (8) can be re-applied to find the solution at any interior point.



Although this discussion was only for Laplace's equation, the boundary element scheme can be generated for many other equations that model a large variety of problems of interest. Using weighted residual schemes, a general method can be developed for many different partial differential equations.