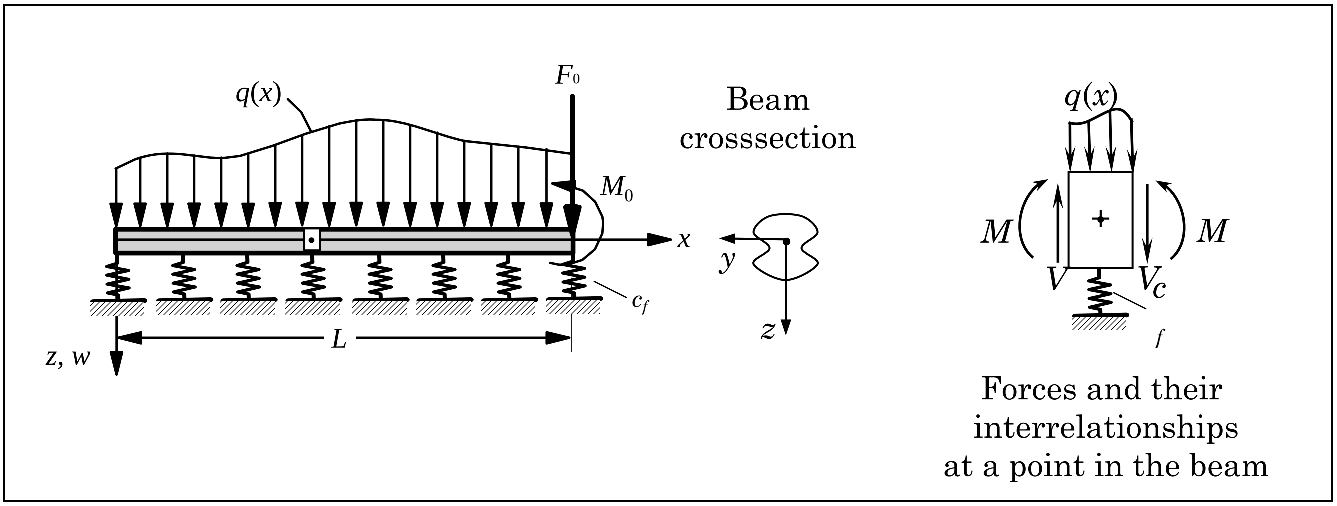
Euler-Bernoulli Beam Finite Element



Definitions of Stress Resultants

$$M = \int_A z \cdot \sigma_{xx} \, dA, \quad V = \int_A \sigma_{xz} \, dA$$

Equilibrium Equations

$$-\frac{dV}{dx} + c_f w = q, \quad \frac{dM}{dx} - V = 0 \;\rightarrow\; -\frac{d^2M}{dx^2} + c_f w = q$$

Kinematic Relations

$$u(x,z) = -z\frac{dw}{dx}, \quad v = 0, \quad w(x,z) = w(x)$$

$$\varepsilon_{xx} = -z\frac{d^2w}{dx^2}, \quad \gamma_{xz} = 0$$

Constitutive Relations

$$\sigma_{xx} = E\varepsilon_{xx} = -Ez\frac{d^2w}{dx^2}, \quad \sigma_{xz} = G\gamma_{xz} = 0$$

Moment-Deflection and Shear Force-Deflection Relations

$$M = \int_A z \cdot \sigma_{xx}\, dA = -EI\frac{d^2w}{dx^2}, \;\; V = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right.$$

Governing Equilibrium Equation in terms of w

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + c_f w = q(x) \quad \text{for} \quad 0 < x < L$$

Weak Form

$$0 = \int_{x_a}^{x_b}\left(EI\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} + c_f vw - vq\right)\, dx - v(x_a)Q_1^e$$
$$- \left(-\frac{dv}{dx}\right)\bigg|_{x_a} Q_2^e - v(x_b)Q_3^e - \left(-\frac{dv}{dx}\right)\bigg|_{x_b} Q_4^e$$

$$Q_1^e \equiv \left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\bigg|_{x_a} = -V(x_a)$$

$$Q_2^e \equiv \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x_e} = -M(x_a)$$

$$Q_3^e \equiv -\left[\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\right]\bigg|_{x_b} = V(x_b)$$

$$Q_4^e \equiv -\left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x_b} = M(x_b)$$

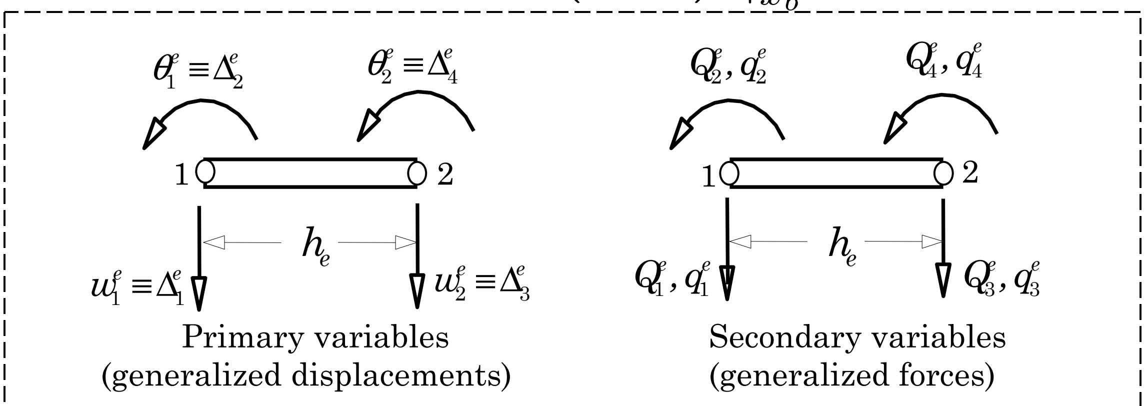
Weak Form (In terms of the bilinear and linear form)

$$0 = B(v, w) - l(v)$$

$$B(v, w) = \int_{x_a}^{x_b} \left(EI \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} + c_f v w \right) dx$$

$$l(v) = \int_{x_a}^{x_b} vq \; dx + v(x_a) Q_1^e + \left(-\frac{dv}{dx} \right) \Big|_{x_a} Q_2^e$$

$$+ v(x_b) Q_3^e + \left(-\frac{dv}{dx} \right) \Big|_{x_b} Q_4^e$$



Finite Element Approximation of w

$$w(x) \approx w_h^e(x) = c_1^e + c_2^e x + c_3^e x^2 + c_4^e x^3$$

$$\Delta_1^e \equiv w_h^e(x_a), \quad \Delta_2^e \equiv -\frac{dw_h^e}{dx}\Big|_{x=x_a},$$

$$\Delta_3^e \equiv w_h^e(x_b), \quad \Delta_4^e \equiv -\frac{dw_h^e}{dx}\Big|_{x=x_b}$$

$$
\begin{aligned}
\Delta_1^e &= w_h^e(x_a) && = c_1^e + c_2^e x_a + c_3^e x_a^2 + c_4^e x_a^3 \\
\Delta_2^e &= -\left.\frac{dw_h^e}{dx}\right|_{x=x_e} && = -c_2^e - 2c_3^e x_a - 3c_4^e x_a \\
\Delta_3^e &= w_h^e(x_b) && = c_1^e + c_2^e x_b + c_3^e x_b^2 + c_4^e x_b^3 \\
\Delta_4^e &= -\left.\frac{dw_h^e}{dx}\right|_{x=x_b} && = -c_2^e - 2c_3^e x_b - 3c_4^e x_b
\end{aligned}
$$

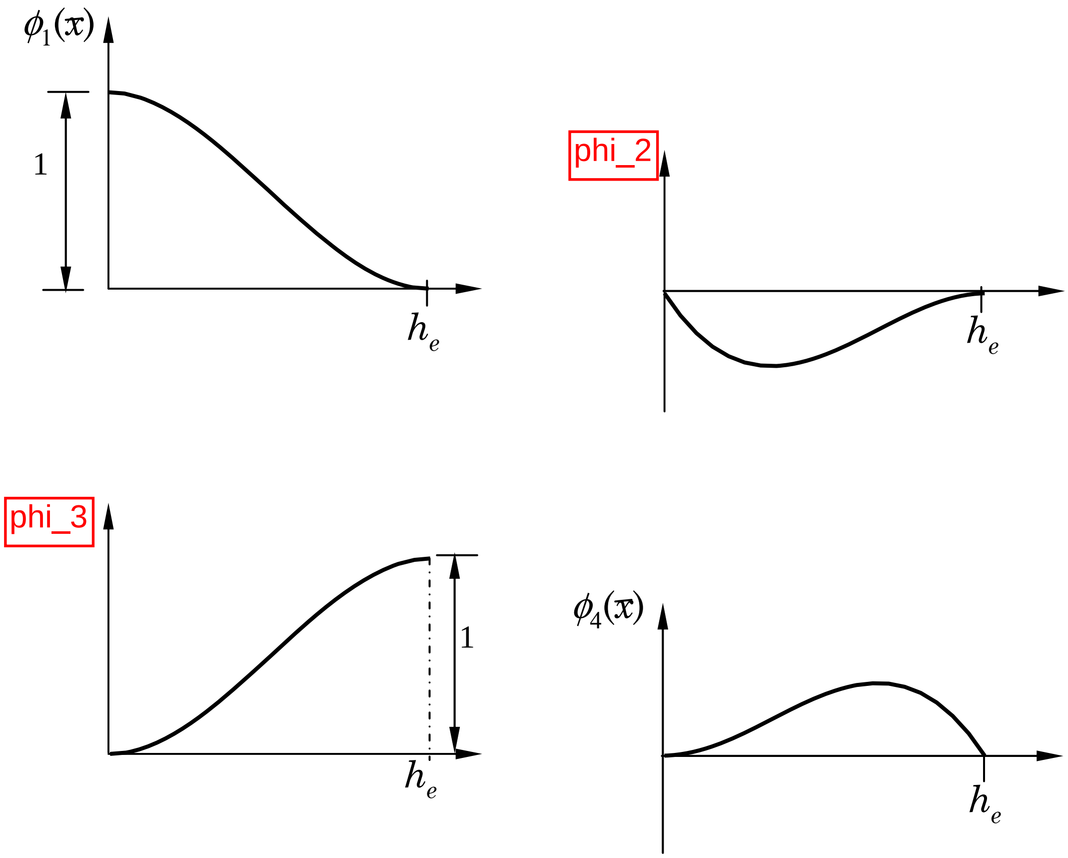
or

$$
\begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{bmatrix} 1 & x_e & x_e^2 & x_e^3 \\ 0 & -1 & -2x_e & -3x_e^2 \\ 1 & x_b & x_b^2 & x_b^3 \\ 0 & -1 & -2x_b & -3x_b^2 \end{bmatrix} \begin{Bmatrix} c_1^e \\ c_2^e \\ c_3^e \\ c_4^e \end{Bmatrix}
$$

$$
w_h^e(x) = \Delta_1^e\phi_1^e + \Delta_2^e\phi_2^e + \Delta_3^e\phi_3^e + \Delta_4^e\phi_4^e = \sum_{j=1}^{4} \Delta_j^e\phi_j^e
$$

$$
\begin{aligned}
\phi_1^e &= 1 - 3\left(\frac{x - x_a}{h_e}\right)^2 + 2\left(\frac{x - x_a}{h_e}\right)^3 \\
\phi_2^e &= -(x - x_a)\left(1 - \frac{x - x_a}{h_e}\right)^2 \\
\phi_3^e &= 3\left(\frac{x - x_a}{h_e}\right)^2 - 2\left(\frac{x - x_a}{h_e}\right)^3 \\
\phi_4^e &= -(x - x_a)\left[\left(\frac{x - x_a}{h_e}\right)^2 - \frac{x - x_a}{h_e}\right]
\end{aligned}
$$

Note that the cubic interpolation functions are derived by interpolating w as well as its derivative dw/dx at the nodes. Such polynomials are known as the <u>Hermite family of interpolation functions,</u> and ϕ_i^e are called the Hermite cubic interpolation (or cubic spline) functions.



Finite Element Equations

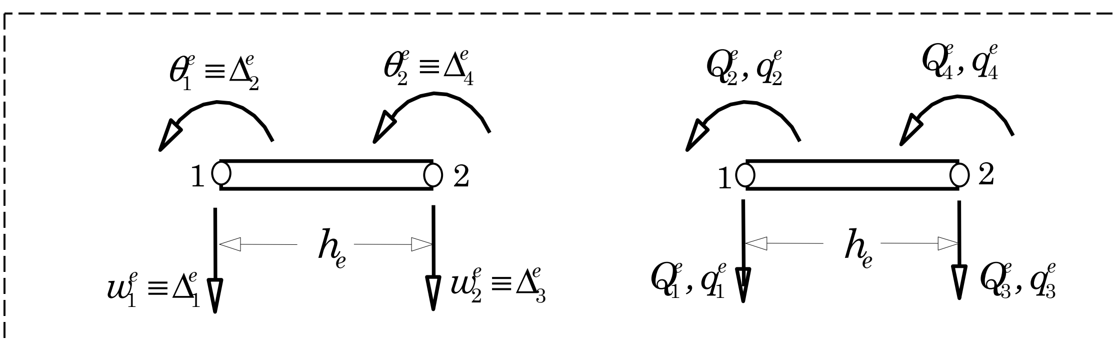
$$0 = \sum_{j=1}^{4} \left[\int_{x_a}^{x_b} \left(EI \frac{d^2 \phi_i^e}{dx^2} \frac{d^2 \phi_j^e}{dx^2} + c_f \, \phi_i^e \phi_j^e \right) dx \right] u_j^e - \int_{x_a}^{x_b} \phi_i^e q \; dx - Q_i^e$$

$$\sum_{j=1}^{4} K_{ij}^e \Delta_j^e - F_i^e = 0 \quad \text{or} \quad [K^e]\{\Delta^e\} = \{F^e\}$$

$$\begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \begin{Bmatrix} \Delta_1^e \\ \Delta_2^e \\ \Delta_3^e \\ \Delta_4^e \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \\ q_3^e \\ q_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(EI \frac{d^2 \phi_i^e}{dx^2} \frac{d^2 \phi_j^e}{dx^2} + c_f \, \phi_i^e \phi_j^e \right) dx$$

$$F_i^e = \int_{x_a}^{x_b} \phi_i^e q \; dx + Q_i^e$$



For element-wise constant values of E_eI_e and q_e:

$$[K^e] = \frac{2E_eI_e}{h_e^3}\begin{bmatrix} 6 & -3h_e & -6 & -3h_e \\ -3h_e & 2h_e^2 & 3h_e & h_e^2 \\ -6 & 3h_e & 6 & 3h_e \\ -3h_e & h_e^2 & 3h_e & 2h_e^2 \end{bmatrix}$$

$$+\frac{c_f^e h_e}{420}\begin{bmatrix} 156 & -22h_e & 54 & 13h_e \\ -22h_e & 4h_e^2 & -13h_e & -3h_e^2 \\ 54 & -13h_e & 156 & 22h_e \\ 13h & -3h_e^2 & 22h_e & 4h_e^2 \end{bmatrix}$$

$$\{F^e\} = \frac{q_e h_e}{12}\begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

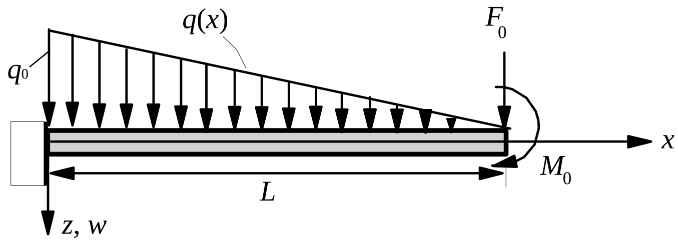
Postprocessing

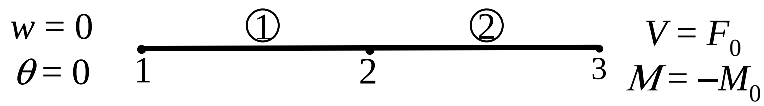
$$M(x) = -EI\frac{d^2w}{dx^2} = -EI\sum_{j=1}^{4}\Delta_j^e\frac{d^2\phi_j^e}{dx^2}$$

$$V(x) = \frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = -EI\sum_{j=1}^{4}\Delta_j^e\frac{d^3\phi_j^e}{dx^3}$$

$$\sigma_x(x,z) = -\frac{M(x)z}{I} = Ez\frac{d^2w}{dx^2} = Ez\sum_{j=1}^{4}\Delta_j^e\frac{d^2\phi_j^e(x)}{dx^2}$$

Example 1





$$q_i^e = \int_{x_a}^{x_b} q_0 \left(1 - \frac{x}{L}\right) \varphi_i^e(x)\ dx$$

$$= \int_0^{h_e} q_0 \left(1 - \frac{\bar{x} + x_a}{L}\right) \varphi_i^e(\bar{x})\ d\bar{x}$$

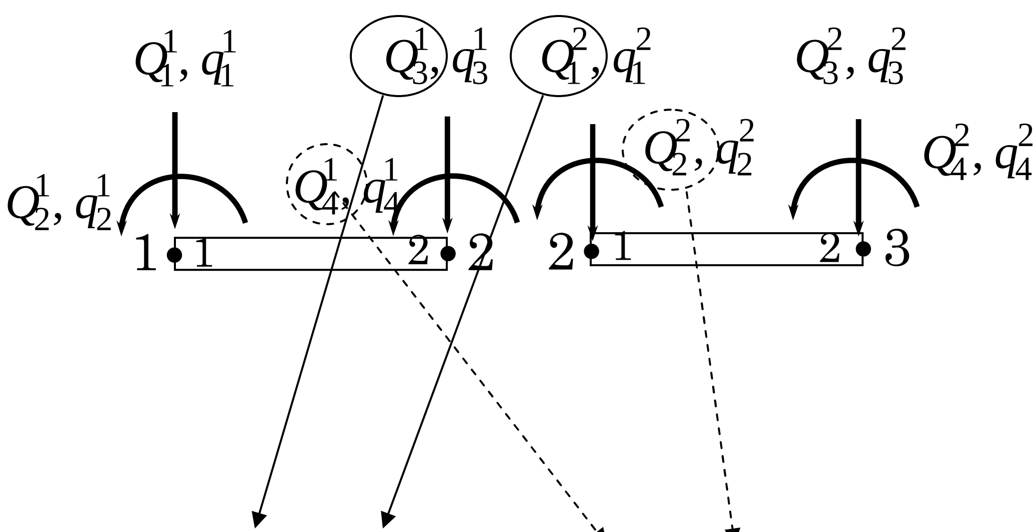
$$\{q^e\} = \frac{q_0 h_e}{12} \begin{Bmatrix} 6 \\ -h_e \\ 6 \\ h_e \end{Bmatrix} + \frac{q_0 h_e}{60L} \begin{Bmatrix} -(9h_e + 30x_a) \\ h_e(2h_e + 5x_a) \\ -(21h_e + 30x_a) \\ -h_e(3h_e + 5x_a) \end{Bmatrix}$$

$$\{q^1\} = \frac{q_0 L}{48} \begin{Bmatrix} 12 \\ -L \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L^2}{480} \begin{Bmatrix} -18 \\ 2L \\ -42 \\ -3L \end{Bmatrix}$$

$$\{q^2\} = \frac{q_0 L}{48} \begin{Bmatrix} 12 \\ -L \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L^2}{480} \begin{Bmatrix} -78 \\ 7L \\ -102 \\ -8L \end{Bmatrix}$$

$$
\frac{2EI}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h & 0 & 0 \\ -3h & 2h^2 & 3h & h^2 & 0 & 0 \\ -6 & 3h & 6+6 & 3h-3h & -6 & -3h \\ -3h & h^2 & 3h-3h & 2h^2+2h^2 & 3h & h^2 \\ 0 & 0 & -6 & 3h & 6 & 3h \\ 0 & 0 & -3h & h^2 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}
$$

$$
= \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L}{480}\begin{Bmatrix} -18 \\ 2L \\ -120 \\ 4L \\ -102 \\ -8L \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_4^1 + Q_2^2 \\ Q_3^2 \\ Q_4^2 \end{Bmatrix}
$$



$$
Q_3^1 + Q_1^2 = 0, \quad Q_4^1 + Q_2^2 = 0
$$

$$
U_1 = U_2 = 0; \quad Q_3^2 = F_0, \quad Q_4^2 = -M_0
$$

Assembled Equations with Boundary Conditions Imposed

$$\frac{4EI}{L^3}\begin{bmatrix} 24 & -6L & -24 & -6L & 0 & 0 \\ -6L & 2L^2 & 6L & L^2 & 0 & 0 \\ -24 & 6L & 48 & 0 & -24 & -6L \\ -6L & L^2 & 0 & 4L^2 & 6L & L^2 \\ 0 & 0 & -24 & 6L & 24 & 6L \\ 0 & 0 & -6L & L^2 & 6L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \frac{q_0 L}{48}\begin{Bmatrix} 12 \\ -L \\ 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L}{480}\begin{Bmatrix} -18 \\ 2L \\ -120 \\ 4L \\ -102 \\ -8L \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ 0 \\ 0 \\ F_0 \\ -M_0 \end{Bmatrix}$$

Condensed Equations for the Displacements

$$\frac{4EI}{L^3}\begin{bmatrix} 48 & 0 & -24 & -6L \\ 0 & 4L^2 & 6L & L^2 \\ -24 & 6L & 24 & 6L \\ -6L & L^2 & 6L & 2L^2 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \frac{q_0 L}{48}\begin{Bmatrix} 24 \\ 0 \\ 12 \\ L \end{Bmatrix} + \frac{q_0 L}{480}\begin{Bmatrix} -120 \\ 4L \\ -102 \\ -8L \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ F_0 \\ -M_0 \end{Bmatrix}$$

Condensed Equations for Generalized Forces

$$\left\{\begin{array}{c} Q_1^1 \\ Q_2^1 \end{array}\right\} = \frac{4EI}{L^3}\begin{bmatrix} -24 & -6L & 0 & 0 \\ 6L & L^2 & 0 & 0 \end{bmatrix}\left\{\begin{array}{c} U_3 \\ U_4 \\ U_5 \\ U_6 \end{array}\right\}$$

$$-\frac{q_0 L}{48}\left\{\begin{array}{c} 12 \\ -L \end{array}\right\} - \frac{q_0 L}{480}\left\{\begin{array}{c} -18 \\ 2L \end{array}\right\}$$

Check Equilibrium of Forces

$$Q_1^1+F_0+\frac{1}{2}q_0L = 0, \quad Q_2^1-(F_0L+\frac{1}{6}q_0L^2+M_0) = 0$$

Generalized Forces using Definitions

$$\begin{aligned}
(Q_1^1)_{\text{def}} &\equiv \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=0} \\
&= EI\left(U_3\frac{d^3\phi_3^1}{dx^3} + U_4\frac{d^3\phi_4^1}{dx^3}\right)\bigg|_{x=0} \\
&= EI\left[U_3\left(-\frac{96}{L^3}\right) + U_4\left(-\frac{24}{L^2}\right)\right] \\
&= -(F_0 + \frac{23}{80}q_0L)
\end{aligned}$$

$$\begin{aligned}
(Q_2^1)_{\text{def}} &\equiv \left(EI\frac{d^2w}{dx^2}\right)\bigg|_{x=0} \\
&= EI\left(U_3\frac{24}{L^2} + U_4\frac{4}{L}\right) = (M_0 + F_0L + \frac{3}{20}q_0L^2)
\end{aligned}$$