Example 3.1

The Saint Venant torsion of a prismatic beam is governed by a differential equation of the type

$$\frac{\partial}{\partial x}\left(\frac{1}{G}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{G}\frac{\partial u}{\partial y}\right) = -2\theta \tag{a}$$

where G is the shear modulus and θ is the rate of twist. The problem variable is the stress function u, such that

$$\tau_{xz} = \frac{\partial u}{\partial y}, \qquad \tau_{yz} = -\frac{\partial u}{\partial x} \tag{b}$$

τ's are the shear stresses. u must have a constant value on the boundary. For convenience it is normally assumed that such constant is equal to zero. Considering an homogeneous material we can write,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 2 \tag{c}$$

where $v = u/G\theta$.

This problem can be solved for v, but noticing that the rate of twist θ is so far unknown. We know that the torque, M_t is

$$M_t = JG\theta = 2\int_{\Omega} u \, \mathrm{d}x\mathrm{d}y \tag{d}$$

where J is the torsional rigidity. We can therefore compute J by evaluating the following integral:

$$J = 2\int_{\Omega} v \, \mathrm{d}x\mathrm{d}y \tag{e}$$

The rate of twist is then given by

$$\theta = M_t/GJ \tag{f}$$

Finally we can compute the shear stresses as

$$\tau_{xz} = G\theta\frac{\partial v}{\partial y}, \qquad \tau_{yz} = -G\theta\frac{\partial v}{\partial x} \tag{g}$$

Consider as an illustration the problem of the torsion of a prismatic bar of elliptical cross section, defined by the equation,

$$\frac{x^2}{a}+\frac{y^2}{b} = 1 \tag{h}$$

For this example we take $a = 2$ and $b = 1$. A finite element mesh selected, consisting of 33 nodes and 48 linear elements is shown in Figure 3.2.

The same example was solved using linear boundary elements (Figure 3.3) with 16 nodes and a series of internal points coinciding

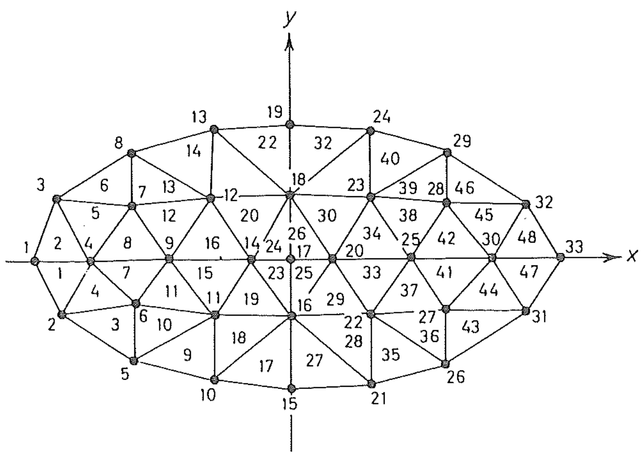

Figure 3.2 Discretisation of the elliptical section using finite elements

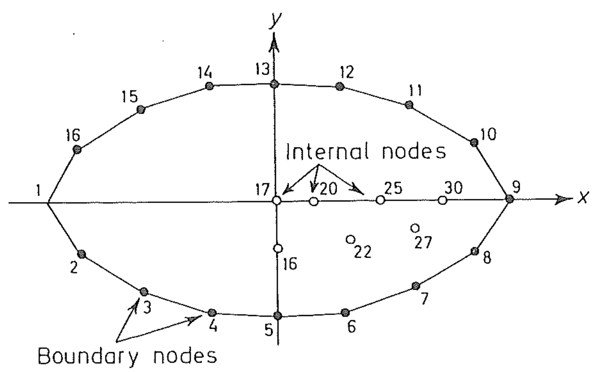

Figure 3.3 Discretisation of the section using boundary elements

with the internal finite element nodes.

The analytical solution for this problem gives $J = 5{\cdot}026$ for the torsional rigidity.

Taking $G = \theta = 1$, one can compute

$$M_t = G\theta J = 5{\cdot}026 \qquad \text{(i)}$$

Table 3.1 includes a comparison between analytical and computed values for the problem variable and the torsional rigidities.

Having the value of the integral of the problem variable over the area of the ellipse, one can compute the approximate value of the torsional rigidity J obtaining

$$J = 2\int_{\Omega} \upsilon \, d\Omega = \begin{cases} 4{\cdot}560 & \text{for finite elements} \\ 4{\cdot}487 & \text{for boundary elements} \end{cases}$$

The approximate solutions are about 10 % of the analytical values. Notice that better agreement is obtained for the problem variable than for the torsional rigidity, which is computed approximately from the approximate values of the problem variables.

Table 3.1 VALUE OF TORSIONAL RIGIDITY AND υ FUNCTION FOR INTERNAL POINTS

Y	*Exact*	*Finite elements (linear)*	*Boundary elements (linear)*
0	0·350	0·341	0·334
−0·35	0·414	0·392	0·401
0	0·638	0·627	0·626
−0·75	0·566	0·561	0·557
0	0·782	0·790	0·772
−0·75	0·638	0·665	0·629
0	0·800	0·793	0·791
Rigidity	5·026	4·560	4·487