Axisymmetric Problems

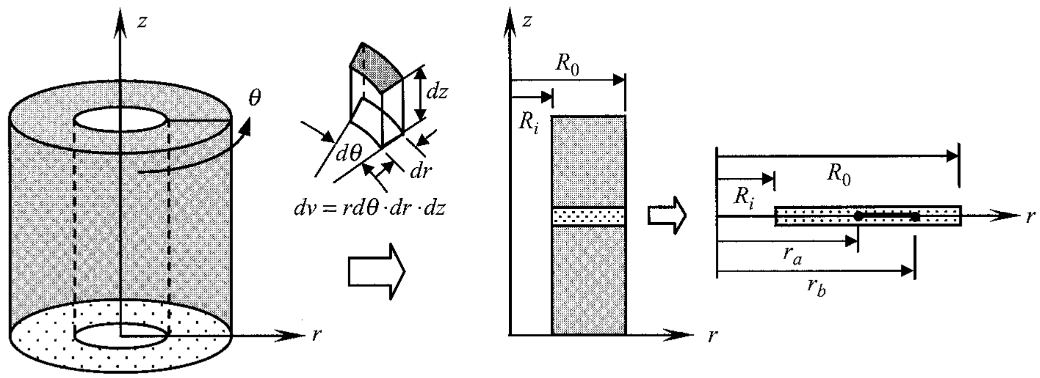

Figure 3.4.1 Volume element and computational domain of an axisymmetric problem.

the radial coordinate r if the problem geometry and data are independent of z. Here, we consider a model second-order equation in a single variable and formulate its finite element model.

Consider the differential equation [an analogue of (3.2.1)]

$$-\frac{1}{r}\frac{d}{dr}\left[a(r)\frac{du}{dr}\right] = f(r) \quad \text{for} \quad R_i < r < R_0 \tag{3.4.1}$$

where r is the radial coordinate, a and f are known functions of r, and u is the dependent variable. Such equations arise, for example, in connection with radial heat flow in a long circular cylinder of inner radius R_i and outer radius R_0. The radially symmetric conditions require that both $a = kr$ (k is the conductivity) and f (internal heat generation) be functions of only r. Since the cylinder is long, the temperature distribution at any section along its length (except perhaps at the ends) is the same, and it is sufficient to consider any cross section away from the ends, i.e., the problem is reduced from a three-dimensional problem to a two-dimensional one. Since a and f are independent of the circumferential direction θ, the temperature distribution along any radial line is the same, reducing the two-dimensional problem to a one-dimensional one, as described by (3.4.1).

3.4.2 Weak Form

In developing the weak form of (3.4.1), we multiply it with a weight function $w(r)$ and integrate over the volume of the cylinder of unit length (see Fig. 3.4.1)

$$\begin{aligned} 0 &= \int_V w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] dv \\ &= \int_0^1 \int_0^{2\pi} \int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] r\,dr\,d\theta\,dz \\ &= 2\pi \int_{r_a}^{r_b} w\left[-\frac{1}{r}\frac{d}{dr}\left(a\frac{du}{dr}\right) - f\right] r\,dr \end{aligned} \tag{3.4.2}$$

where (r_a, r_b) is the domain of a typical element along the radial direction. Next, we carry out the remaining two steps of the weak formulation:

$$0 = 2\pi \int_{r_a}^{r_b} \left(a \frac{dw}{dr} \frac{du}{dr} - rwf \right) dr - 2\pi \left[wa \frac{du}{dr} \right]_{r_a}^{r_b}$$

$$0 = 2\pi \int_{r_a}^{r_b} \left(a \frac{dw}{dr} \frac{du}{dr} - rwf \right) dr - w(r_a) Q_1^e - w(r_b) Q_2^e \tag{3.4.3a}$$

where

$$Q_1^e \equiv -2\pi \left(a \frac{du}{dr} \right) \bigg|_{r_a}, \quad Q_2^e \equiv 2\pi \left(a \frac{du}{dr} \right) \bigg|_{r_b} \tag{3.4.3b}$$

3.4.3 Finite Element Model

The finite element model is obtained by substituting the approximation

$$u(r) \approx \sum_{j=1}^{n} u_j^e \psi_j^e(r) \tag{3.4.4}$$

and $w = \psi_1, \psi_2, \ldots, \psi_n$ into (3.4.3a). The finite-element model is given by

$$[K^e]\{u^e\} = \{f^e\} + \{Q^e\} \tag{3.4.5a}$$

where

$$K_{ij}^e = 2\pi \int_{r_a}^{r_b} a \frac{d\psi_i^e}{dr} \frac{d\psi_j^e}{dr} dr, \quad f_i^e = 2\pi \int_{r_a}^{r_b} \psi_i^e fr\, dr \tag{3.4.5b}$$

and ψ_i^e are the interpolation functions expressed in terms of the radial coordinate r. For example, the linear interpolation functions are of the form ($h_e = r_b - r_a$) [see Eq. (3.2.17)]

$$\psi_1^e(r) = \frac{r_b - r}{h_e}, \quad \psi_2^e(r) = \frac{r - r_a}{h_e} \tag{3.4.6}$$

The explicit forms of the coefficients K_{ij}^e and f_i^e for $a = a_e r$ and $f = f_e$ are given below (r_a denotes the global coordinate of node 1 of the element).

Linear Element

$$[K^e] = \frac{2\pi a_e}{h_e}(r_a + \frac{1}{2}h_e) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \{f^e\} = \frac{2\pi f_e h_e}{6} \begin{Bmatrix} 3r_a + h_e \\ 3r_a + 2h_e \end{Bmatrix} \tag{3.4.7}$$

Quadratic Element

$$[K^e] = \frac{2\pi a_e}{6h_e} \begin{bmatrix} 3h_e + 14r_a & -(4h_e + 16r_a) & h_e + 2r_a \\ -(4h_e + 16r_a) & 16h_e + 32r_a & -(12h_e + 16r_a) \\ h_e + 2r_a & -(12h_e + 16r_a) & 11h_e + 14r_a \end{bmatrix}$$

$$\{f^e\} = \frac{2\pi f_e h_e}{6} \begin{Bmatrix} r_a \\ 4r_a + 2h_e \\ r_a + h_e \end{Bmatrix} \tag{3.4.8}$$