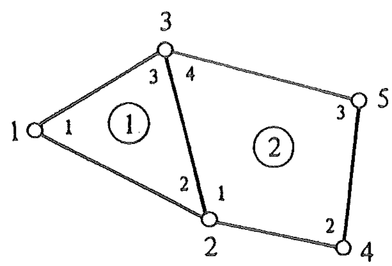
Additional Features of Two-Dimensional Finite Element Analysis

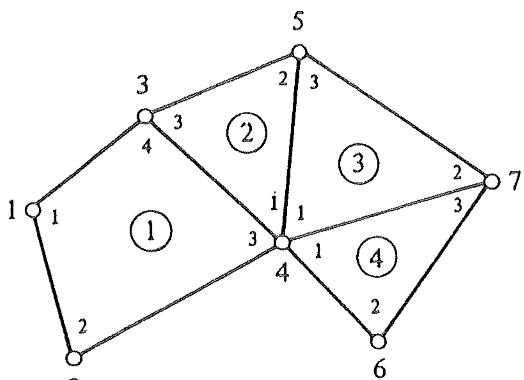
Assembly
Bandwidth
Boundary Conditions
Higher Order Elements

Assembly of Element Matrices



(*a*)

Global →	Local
K_{11}	$K_{11}^{(1)}$
K_{12}	$K_{12}^{(1)}$
K_{22}	$K_{22}^{(1)} + K_{11}^{(2)}$
K_{14}	0
K_{15}	0
K_{23}	$K_{23}^{(1)} + K_{14}^{(2)}$
⋮	



(*b*)

Global →	Local
K_{14}	$K_{13}^{(1)}$
K_{34}	$K_{43}^{(1)} + K_{31}^{(2)}$
K_{17}	0
K_{44}	$K_{33}^{(1)} + K_{11}^{(2)} + K_{11}^{(3)} + K_{11}^{(4)}$
K_{45}	$K_{12}^{(2)} + K_{13}^{(3)}$
K_{56}	0
K_{47}	$K_{12}^{(3)} + K_{13}^{(4)}$
⋮	

(1, 1)	(1, 1) of element 1
(1, 2)	(1, 2) of element 1
(1, 3)	(1, 3) of element 1
(1, 4)	No correspondence
(1, 5)	No correspondence
(2, 2)	(2, 2) of element 1 and (1, 1) of element 2
(2, 3)	(2, 3) of element 1 and (1, 4) of element 2
(2, 4)	(1, 2) of element 2
(2, 5)	(1, 3) of element 2
(3, 3)	(3, 3) of element 1 and (4, 4) of element 2
(3, 4)	(4, 2) of element 2
(3, 5)	(4, 3) of element 2
(4, 4)	(2, 2) of element 2
(4, 5)	(2, 3) of element 2
(5, 5)	(3, 3) of element 2

Labeling of the Nodes
Bandwidth Consideration

The nature of the stiffness matrix will change depending on the numbering scheme used for the nodes. In specific, the so-called *bandwidth*, B will be effect by such numbering as follows

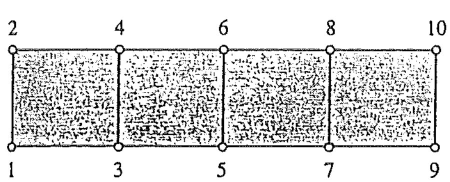
$$B = (R + 1)\ \text{NDOF}$$

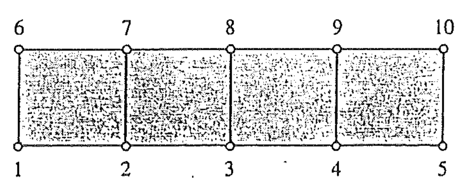
Bandwidth

$$\begin{bmatrix} C & C & C & 0 & C & 0 & 0 & 0 & 0 \\ C & C & C & C & C & C & 0 & 0 & 0 \\ C & C & C & C & 0 & C & C & 0 & 0 \\ 0 & C & C & C & C & C & C & C & 0 \\ C & C & 0 & C & C & C & C & 0 & C \\ 0 & C & C & C & C & C & C & C & C \\ 0 & 0 & C & C & C & C & C & C & 0 \\ 0 & 0 & 0 & C & 0 & C & C & C & C \\ 0 & 0 & 0 & 0 & C & C & 0 & C & C \end{bmatrix}$$

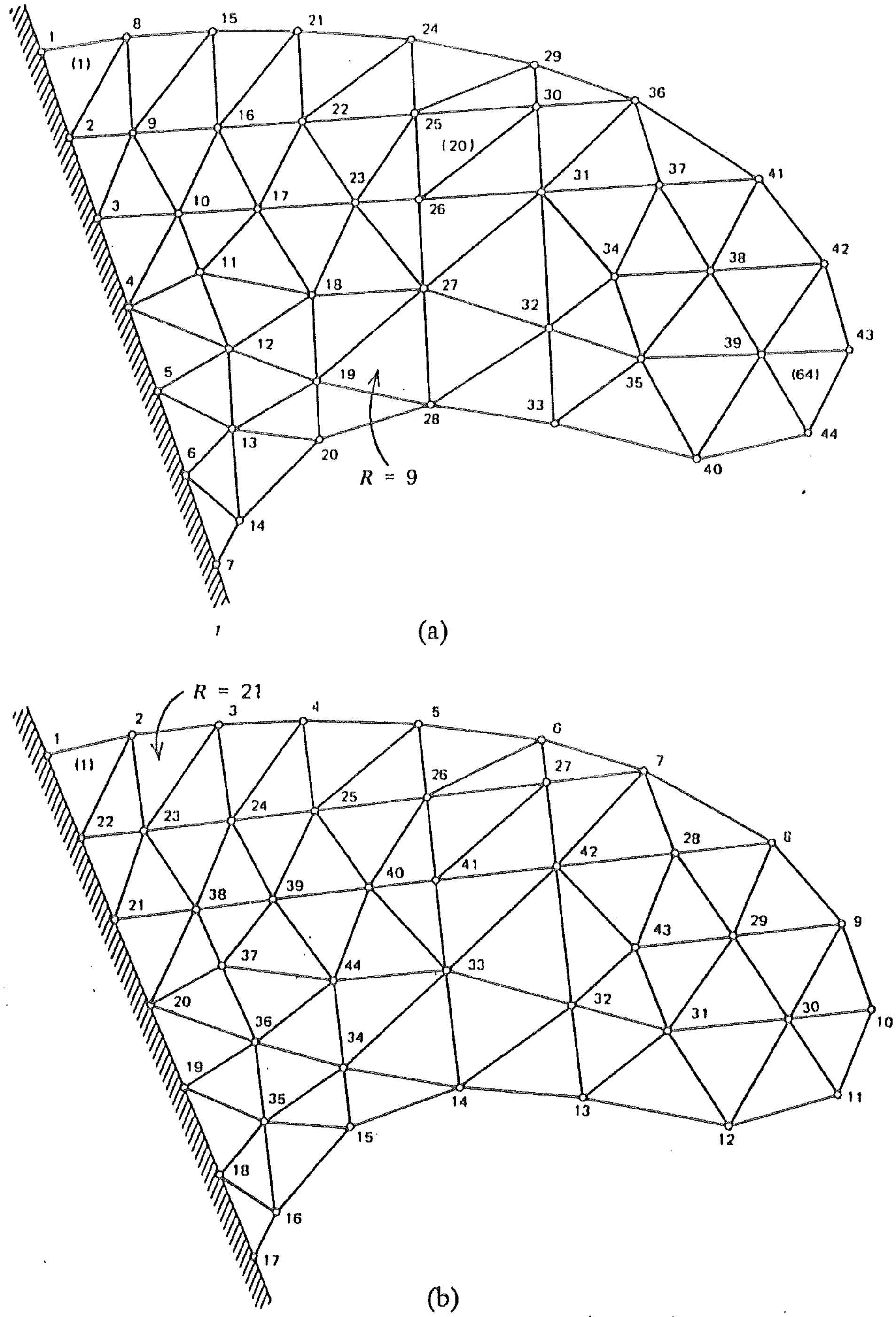
In the formula for B, R is the largest difference between the node numbers in any given element, and NDOF is the number of degrees of freedom at each node.

As an example, consider for a scalar valued problem, the two meshes and their respective stiffness matrices shown below





$$\begin{bmatrix} x & x & x & x & & & & & & \\ x & x & x & x & & & & & & \\ x & x & x & x & x & x & & & & \\ x & x & x & x & x & x & & & & \\ & & x & x & x & x & x & x & & \\ & & x & x & x & x & x & x & & \\ & & & & x & x & x & x & x & x \\ & & & & x & x & x & x & x & x \\ & & & & & & x & x & x & x \\ & & & & & & x & x & x & x \end{bmatrix} \qquad \begin{bmatrix} x & x & & & & x & x & & & \\ x & x & x & & & x & x & x & & \\ & x & x & x & & & x & x & x & \\ & & x & x & x & & & x & x & x \\ & & & x & x & & & & x & x \\ x & x & & & & x & x & & & \\ x & x & x & & & x & x & x & & \\ & x & x & x & & & x & x & x & \\ & & x & x & x & & & x & x & x \\ & & & x & x & & & & x & x \end{bmatrix}$$



Two examples of node numbering for a two-dimensional body.

Imposition of Boundary Conditions

specification of two different values of a primary variable at the same boundary point

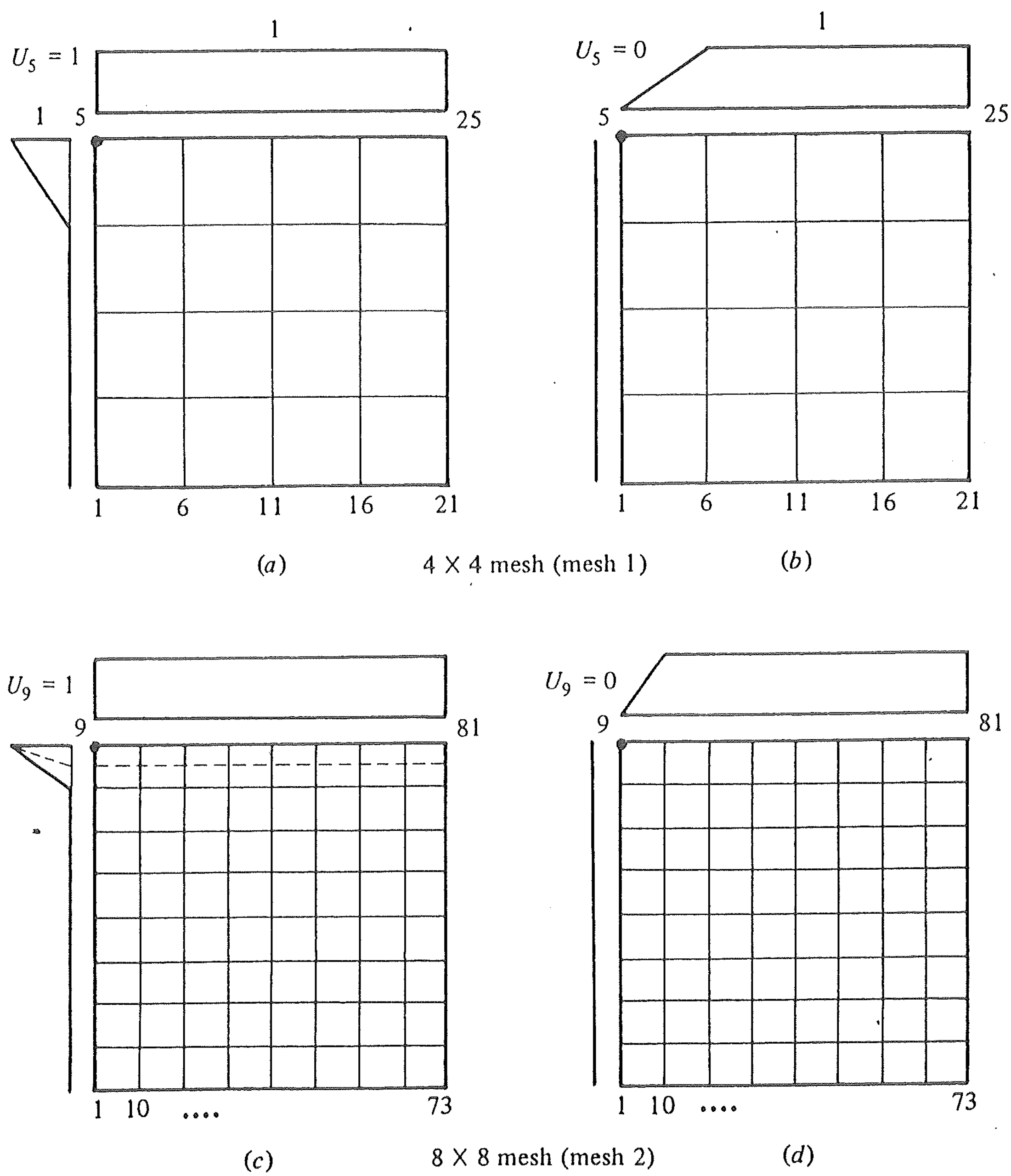


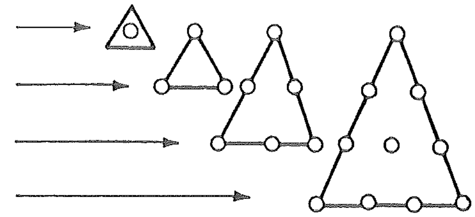
Figure 4.20 Effect of specifying (either of the) two values of a primary variable at a boundary node [node 5 in (*a*) and (*b*) and node 9 in (*c*) and (*d*)].

INTERPOLATION FUNCTIONS

Triangular Elements

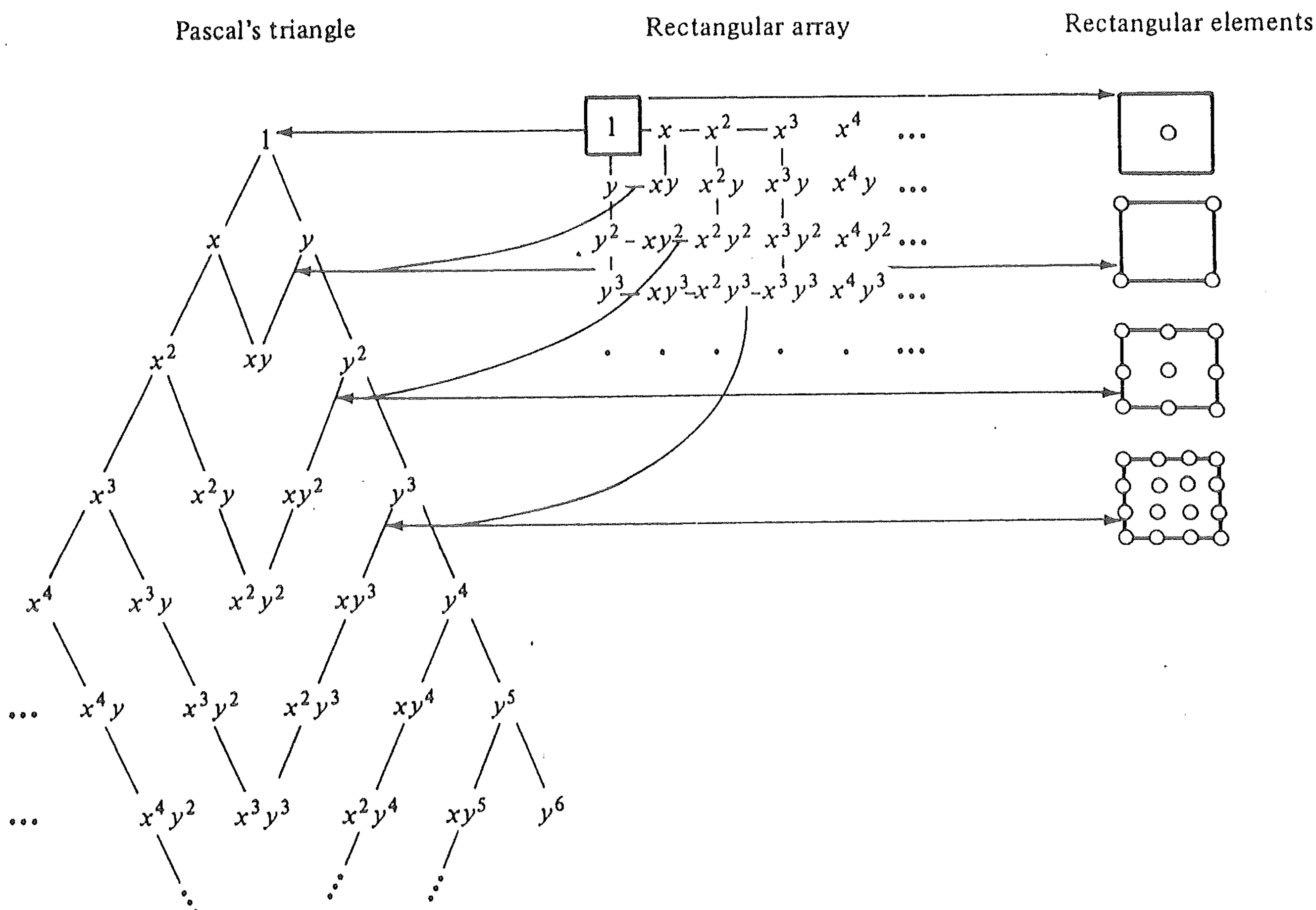
Pascal's triangle	Degree of the polynomial	Number of terms in the polynomial
1	0	1
$x \quad y$	1	3
$x^2 \quad xy \quad y^2$	2	6
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3	10
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4	15
$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$	5	21
$x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6$	6	28
. . . .	⋮	⋮

Element with nodes



Pascal's triangle for the generation of the Lagrange family of triangular elements.

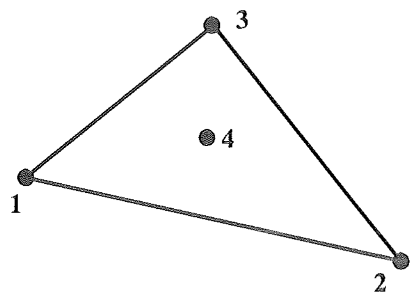
Rectangular Elements



Lagrange family of rectangular elements of various order.

Condensation of Internal Nodes

Internal nodes of the higher-order elements of the Lagrange family do not contribute to the interelement connectivity. Hence they can be*condensed* out of the problem at the element level so that the size of the element matices does not become excessively large. Consider, as an example, the triangluar element with one internal node as shown.



The element equation for this case would read

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ \cdot & K_{22} & K_{23} & K_{24} \\ \cdot & \cdot & K_{33} & K_{34} \\ \cdot & \cdot & \cdot & K_{44} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

This equation can be written in the equivalent form

$$\begin{bmatrix} [K'] & \{K''\} \\ \{K''\}^T & K_{44} \end{bmatrix} \begin{pmatrix} \{u\} \\ u_4 \end{pmatrix} = \begin{pmatrix} \{F'\} \\ F_4 \end{pmatrix}$$

This system of equations can thus be separated, and solving the last equation for u_4 yields

$$u_4 = \frac{1}{K_{44}} \left(F_4 - \{K''\}^T\{u\}\right)$$

Using this result back into the first partioned system, yields

$$[K']\{u\} + \frac{\{K''\}}{K_{44}}\left(F_4 - \{K''\}^T\{u\}\right) = \{F'\}$$

which can be re-written in the standard form

$$[\hat{K}]\ \{u\} = \{\hat{F}\}$$

with appropriate definitions for the new stiffness and column matrices. Thus this new element equation now contains only the boundary nodal unknowns. This procedure can be carried out for any number of internal nodes.