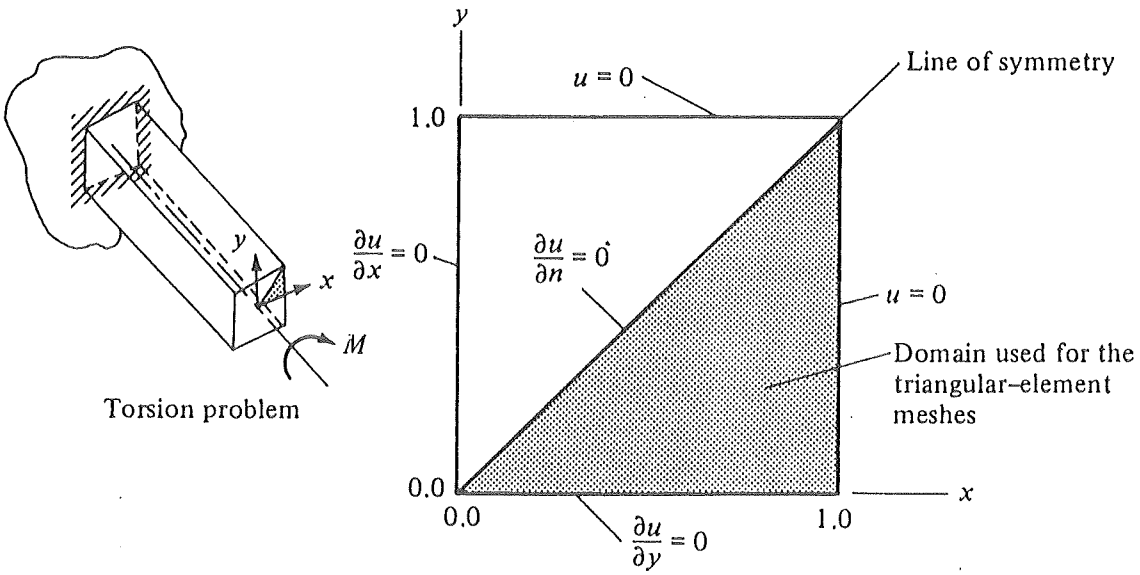
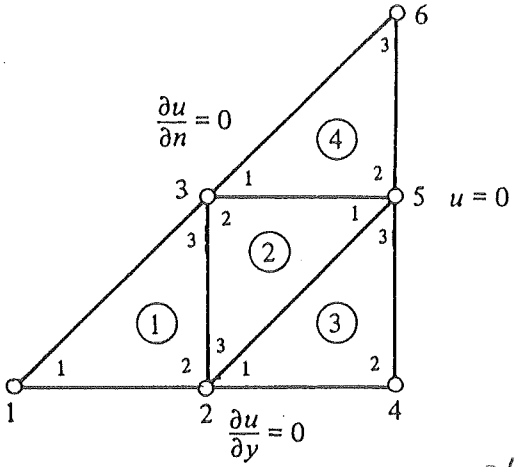
Example Poisson equation

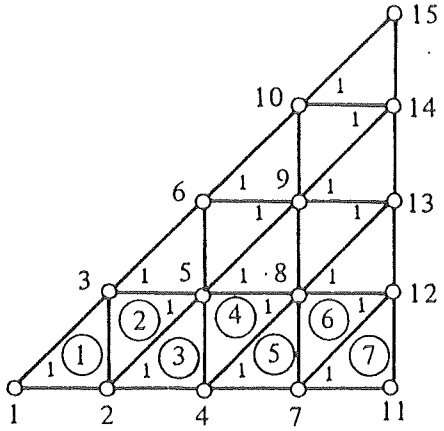
$$-\nabla^2 u = 1 \quad \text{in } \Omega = \{(x, y): \ 0 < (x, y) < 1\}$$

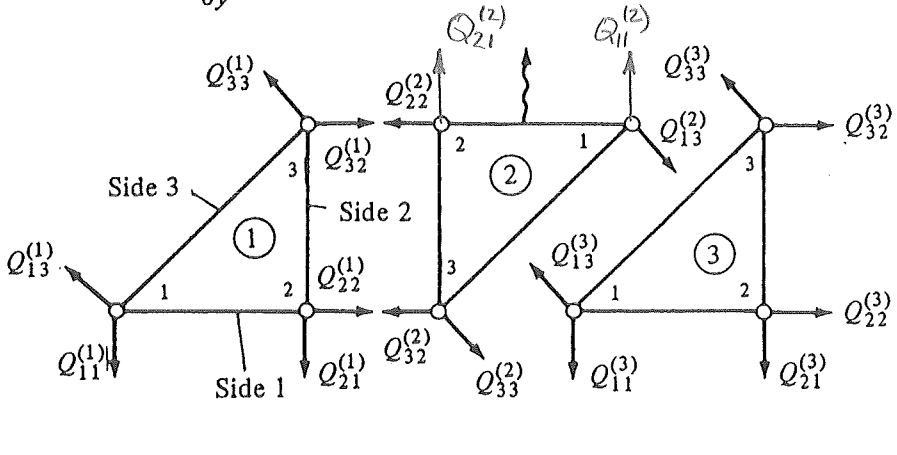
$$\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial y}(x, 0) = 0 \qquad u(1, y) = u(x, 1) = 0$$

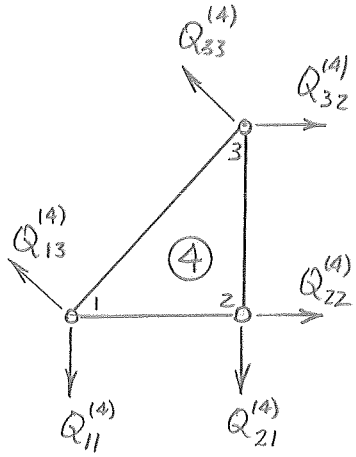


(a)









Solution by linear triangular elements

$$K_{ij}^{(e)} = \frac{1}{4A}\left(\beta_i^{(e)}\beta_j^{(e)} + \gamma_i^{(e)}\gamma_j^{(e)}\right) \qquad f_i^{(e)} = \frac{fA}{3}$$

$$[K^{(1)}] = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad \{F^{(1)}\} = \frac{1}{24}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ Q_3^{(1)} \end{Bmatrix}$$

$$\begin{aligned} Q_i^{(1)} &= \oint_{\Gamma^e} q_n^{(1)}\psi_i^{(1)}(s)\, ds \\ &= \int_0^{0.5}\left[q_n^{(1)}\psi_i^{(1)}(x, y)\right]_{y=0} dx + \int_0^{0.5}\left[q_n^{(1)}\psi_i^{(1)}(x, y)\right]_{x=0.5} dy \\ &\quad + \int_{0.5}^{0}\left[q_n^{(1)}\psi_i^{(1)}(x, y)\right]_{x=y} dx \\ &\equiv Q_{i1}^{(1)} + Q_{i2}^{(1)} + Q_{i3}^{(1)} \end{aligned}$$

$$[B] = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 2 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

$$[K] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 & 0 \\ & K_{22}^{(1)} + K_{33}^{(2)} + K_{11}^{(3)} & K_{23}^{(1)} + K_{32}^{(2)} & K_{12}^{(3)} & K_{13}^{(3)} + K_{31}^{(2)} & 0 \\ & & K_{33}^{(1)} + K_{22}^{(2)} + K_{11}^{(4)} & 0 & K_{21}^{(2)} + K_{12}^{(4)} & K_{13}^{(4)} \\ \text{Symmetric} & & & K_{22}^{(3)} & K_{23}^{(3)} & 0 \\ & & & & K_{11}^{(2)} + K_{33}^{(3)} + K_{22}^{(4)} & K_{23}^{(4)} \\ & & & & & K_{33}^{(4)} \end{bmatrix}$$

$$\{F\} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} + F_3^{(2)} + F_1^{(3)} \\ F_3^{(1)} + F_2^{(2)} + F_1^{(4)} \\ F_2^{(3)} \\ F_1^{(2)} + F_3^{(3)} + F_2^{(4)} \\ F_3^{(4)} \end{Bmatrix}$$

assembled system

$$[K^{(1)}] = [K^{(2)}] = [K^{(3)}] = [K^{(4)}]$$

$$\{F^{(1)}\} = \{F^{(2)}\} = \{F^{(3)}\} = \{F^{(4)}\}$$

$$\frac{1}{2}\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 & 0 \\ 0 & -2 & 4 & 0 & -2 & 0 \\ \hdashline 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]\left\{\begin{array}{c} U_1 \\ U_2 \\ U_3 \\ \hdashline U_4 \\ U_5 \\ U_6 \end{array}\right\} = \frac{1}{24}\left\{\begin{array}{c} 1 \\ 3 \\ 3 \\ \hdashline 1 \\ 3 \\ 1 \end{array}\right\}$$

$$+ \left\{\begin{array}{c} Q_1^{(1)} \\ Q_2^{(1)} + Q_3^{(2)} + Q_1^{(3)} \\ Q_3^{(1)} + Q_2^{(2)} + Q_1^{(4)} \\ \hdashline Q_2^{(3)} \\ Q_1^{(2)} + Q_3^{(3)} + Q_2^{(4)} \\ Q_3^{(4)} \end{array}\right\}$$

boundary conditions

$$U_4 = U_5 = U_6 = 0$$

$$Q_1^{(1)} = Q_{11}^{(1)} + Q_{13}^{(1)} = \text{specified to be zero} = 0$$

$$Q_2^{(1)} + Q_3^{(2)} + Q_1^{(3)} = \left(Q_{21}^{(1)} + Q_{11}^{(3)}\right) + \left[\left(Q_{22}^{(1)} + Q_{32}^{(2)}\right) + \left(Q_{13}^{(3)} + Q_{33}^{(2)}\right)\right]$$

$$= \text{specified to be zero} + [\text{cancel to give zero}] = 0$$

$$Q_3^{(1)} + Q_2^{(2)} + Q_1^{(4)} = Q_{33}^{(1)} + Q_{13}^{(4)} = 0$$

$$Q_2^{(3)} = Q_{21}^{(3)} + Q_{22}^{(3)} = \text{specified to be zero} + \text{unknown} = Q_{22}^{(3)}, \text{ unknown}$$

$$Q_1^{(2)} + Q_3^{(3)} + Q_2^{(4)} = Q_{32}^{(3)} + Q_{22}^{(4)}, \text{ unknown}$$

$$Q_3^{(4)} = Q_{32}^{(4)} + Q_{33}^{(4)} = \text{unknown} + \text{zero due to symmetry} = Q_{32}^{(4)}, \text{ unknown}$$

Solution

$$\begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 2.0 & -1.0 \\ 0 & -1.0 & 2.0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{1}{24} \begin{Bmatrix} 1 \\ 3 \\ 3 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_{22}^{(3)} \\ Q_{32}^{(3)} + Q_{22}^{(4)} \\ Q_{32}^{(4)} \end{Bmatrix} = -\frac{1}{24} \begin{Bmatrix} 1 \\ 3 \\ 1 \end{Bmatrix} + \begin{bmatrix} 0 & -0.5 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \frac{1}{24} \begin{bmatrix} 3 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 0.5 & 0.5 & 0.75 \end{bmatrix} \begin{Bmatrix} 1 \\ 3 \\ 3 \end{Bmatrix} = \frac{1}{24} \begin{Bmatrix} 7.5 \\ 5.5 \\ 4.25 \end{Bmatrix} = \begin{Bmatrix} 0.31250 \\ 0.22917 \\ 0.17708 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_{22}^{(3)} \\ Q_{32}^{(3)} + Q_{22}^{(4)} \\ Q_{32}^{(4)} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{24} - 0.5U_2 \\ -\frac{1}{8} - U_3 \\ -\frac{1}{24} \end{Bmatrix} = \begin{Bmatrix} -0.197917 \\ -0.302083 \\ -0.041667 \end{Bmatrix}$$

Table 4.1 Comparison of the finite-element solutions $u(0, y)$ with the series solution and the Ritz solution of Eqs. (4.47)

	Triangular elements		Rectangular elements		Ritz,	Series solution,
y	4 elements	16 elements	4 elements	16 elements	Eq. (2.96)	Eq. (2.95)
0.0	0.3125	0.3013	0.3107	0.2984	0.3125	0.2947
0.25	0.2709†	0.2805	0.2759†	0.2824	0.2930	0.2789
0.50	0.2292	0.2292	0.2411	0.2322	0.2344	0.2293
0.75	0.1146†	0.1393	0.1205†	0.1414	0.1367	0.1397
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

†Interpolated values.

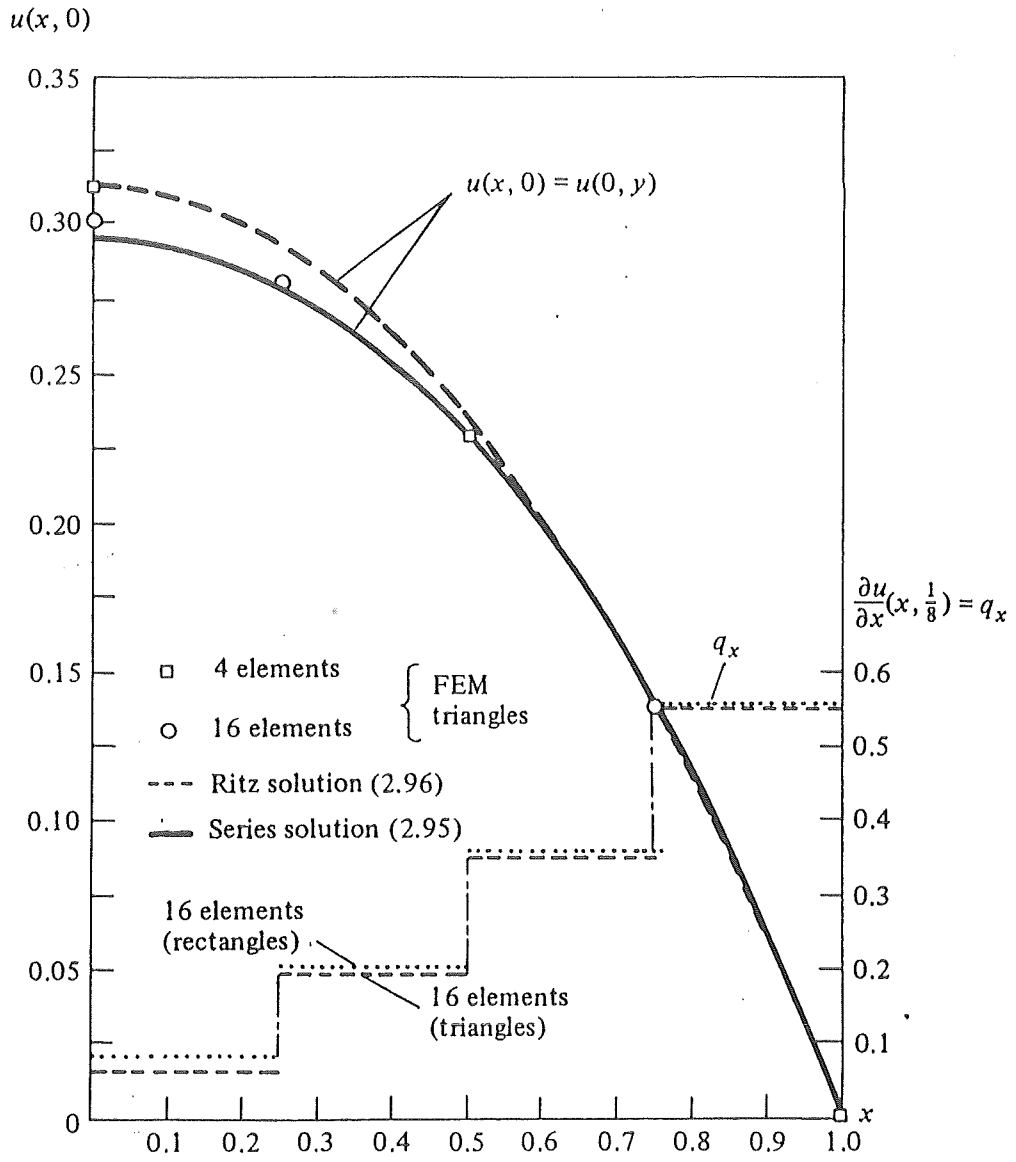
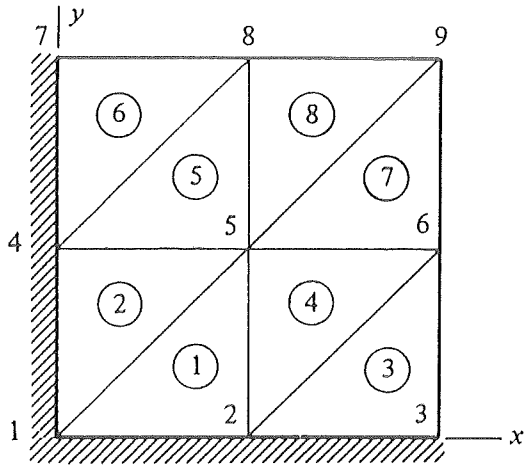


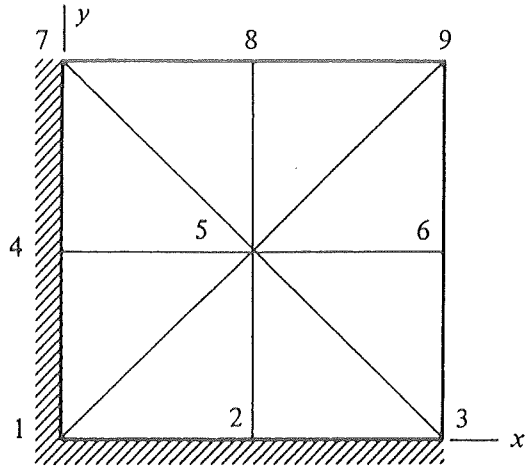
Figure 4.10 Comparison of the finite-element solution with the two-parameter Ritz solution and series solution of Eqs. (4.47).

Table 4.4 Comparison of the finite-element solutions obtained by various triangular-element meshes with the series solution of the problem in Example 4.1

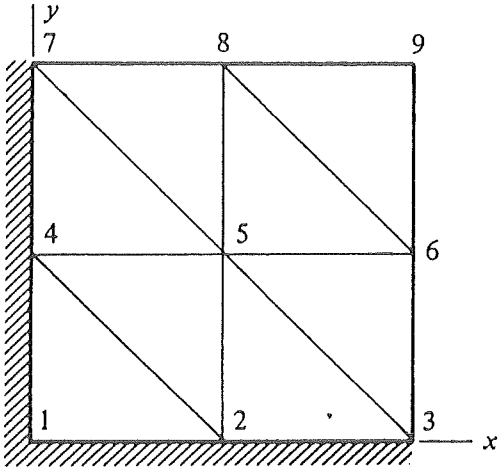
Node	Finite-element solution			Series solution
	Mesh 1	Mesh 2	Mesh 3	
1	0.31250	0.29167	0.25000	0.29469
2	0.22917	0.20833	0.20833	0.22934
4	0.22917	0.20833	0.20833	0.22934
5	0.17708	0.18750	0.16667	0.18114



Mesh 1.



Mesh 2.



Mesh 3.

Table 4.5 Convergence of the finite-element solution (with the mesh refinement†) of the problem in Example 4.1

Location		Finite-element solution				Series solution
x	y	2×2	4×4	6×6	8×8	
0.0	0.0	0.31071	0.29839	0.29641	0.29560	0.29469
0.125	0.0	—	—	0.29248	0.29167	0.29077
0.250	0.0	—	0.28239	0.28055	0.27975	0.27888
0.375	0.0	—	—	0.26022	0.24943	0.25863
0.50	0.0	0.24107	0.23220	0.23081	0.23005	0.22934
0.625	0.0	—	—	—	0.19067	0.19009
0.750	0.0	—	0.14137	0.14064	0.14014	0.13973
0.875	0.0	—	—	—	0.07709	0.07687
0.125	0.125	—	—	0.28862	0.28781	0.28692
0.250	0.250	—	0.26752	0.26580	0.26498	0.26415
0.375	0.375	—	—	0.22960	0.22873	0.22799
0.50	0.50	0.19286	0.18381	0.18282	0.18179	0.18114
0.625	0.625	—	—	—	0.12813	0.12757
0.750	0.750	—	0.07506	0.07481	0.07332	0.07282
0.875	0.875	—	—	—	0.02561	0.02510

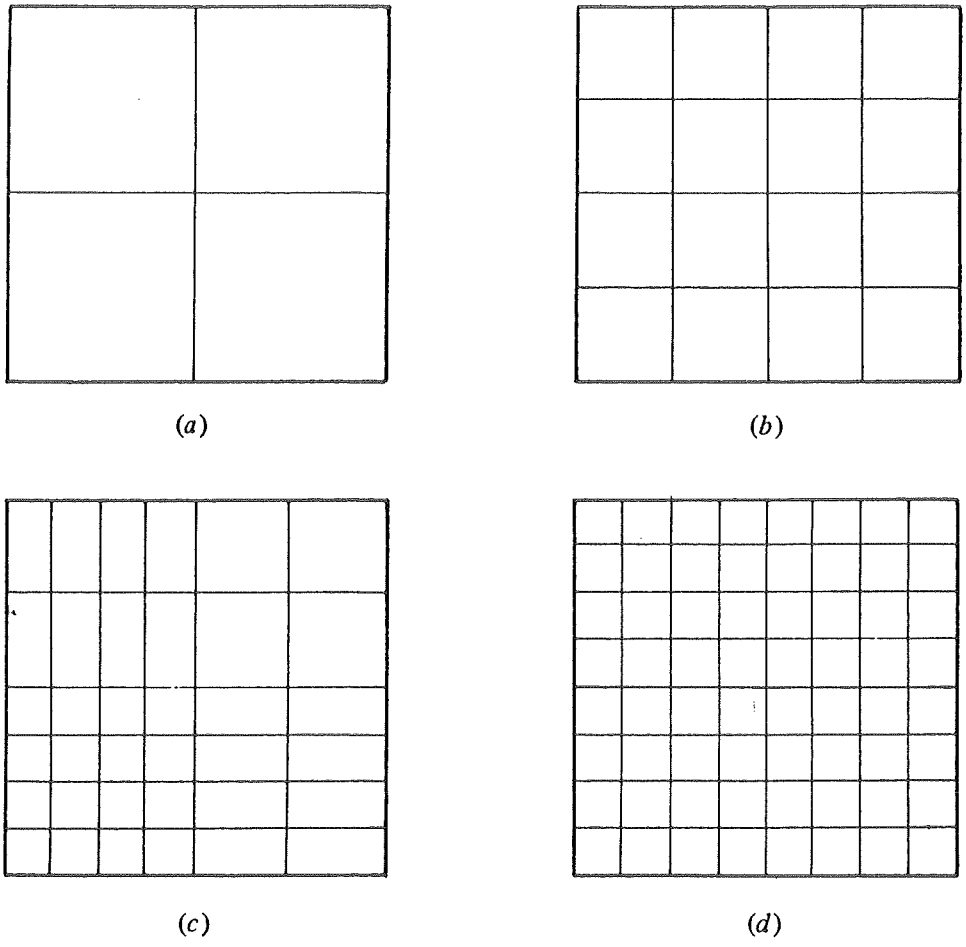


Figure 4.22 Mesh refinement: meshes in (a), (b), and (d) are uniform; mesh in (c) is nonuniform. (a) 2×2 mesh. (b) 4×4 mesh. (c) 6×6 mesh. (d) 8×8 mesh.