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Nonlinear Waves on Steep Slopes

Ib A. Svendsen and Stéphan T. Grilli

Department of Civil Engineering
University of Delaware
Newark, DE 19716, U.S.A.

ABSTRACT

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A high accuracy Boundary Element Method for fully nonlinear waves has been developed which solves the problems in physical space and, hence, can readily deal with boundaries, structures and bottom topographies of arbitrary shape. It is applied to runup and reflection of waves on steep slopes and verified by comparison with surface measurements in laboratory experiments with a solitary wave. Subsequently, the method is used to give a detailed picture of the internal velocity field inside the wave and the pressure along the slope as it develops during uprush and downrush. The relevance and validity of, particularly, models based on the nonlinear shallow water equations are examined.

ADDITIONAL INDEX WORDS: *Fully nonlinear waves, boundary element method, solitary wave runup, nonlinear shallow water equations.*

INTRODUCTION

This paper is presenting detailed computational results for the particle velocities in wave motion on coastal and ocean structures with steep front slopes such as dikes, seawalls and rubble mound breakwaters. Such structures usually have front slopes ranging between 1:1.5 (33:7) and 1:2.5 (21:8) (see BRUUN, 1976) and the method used here is particularly suited for such conditions. It is also discussing the relevance of the method used relative to other approaches to the same problem.

The problem of run-up and overtopping on steep slope structures is not just related to determining their necessary crest height over the mean sea level. It is also of crucial importance for the stability of the armor units protecting the slopes against erosion and failure. This problem developed into one of the central questions in modern coastal engineering when in the 70's a substantial number of rubble mound breakwaters, most of them in the Mediterranean or on the Iberian Peninsula, were severely damaged by storm waves. In the most famous and disastrous case the not yet completed breakwater at Sines in Portugal was almost completely destroyed, an event which delayed the ongoing industrial development of the whole southern part of Portugal and had severe repercussions for the country's economy.

Such large scale disasters attract much attention. More importantly, however, they severely influence the development within the profession and signal that satisfactory methods for the analysis and design of safe and economic rubble mound structures still remain to be developed.

The present paper briefly outlines the ideas behind the Boundary Element Method (BEM), applied to fully nonlinear waves (section 2) and the difference between the present approach and other versions known from the literature (section 3). Results are then presented in section 4 for a comparison between detailed measurements and computations of a solitary wave on a steep slope and in section 5 is given an account of other theoretical methods available for treating the problem with particular emphasis on the non-linear shallow water (NSW) equations. It is found that it is crucial for the validity of the NSW approach that pressure is hydrostatic and horizontal velocities uniform over depth and focuses on analyzing the validity of this assumption using the BEM computational results. In section 6 we compare our results for runup with an analytical solution and finally section 7 gives a critical assessment of the strong and weak points of the theoretical models discussed.

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2. THE METHOD OF COMPUTATION

The method used for obtaining the results described in the following is a so-called Boundary Integral Equation Method, BIEM. Although this method may be developed (at least in the two-dimensional case) to include dissipation effects such as bottom friction, the results here correspond to potential flow. As mentioned in section 8, the effect of actual bottom friction will be very minor on a steep slope unless the roughness is very large. This section discusses the ideas behind the method. For a detailed mathematical description, reference is made to GRILLI *et al.*, (1989).

By suitable mathematical manipulations, the problem of determining the velocity potential (or the stream function) in a flow region of interest can be transformed into an integral equation which only requires consideration of the velocity potential and its normal derivative along the boundary of the flow region. Thus a two dimensional problem is reduced to a problem in one dimension (the boundary curve) and a three dimensional problem to a problem in two dimensions (along the boundary surface). These benefits, however, are partly offset by a more complicated numerical solution technique required. The real advantage of the method, however, is not as much the reduction of computations from two to one (or three to two) dimensions as its flexibility to handle with ease and high numerical accuracy, problems with complicated boundary geometries. In water waves, this flexibility has been utilized to problems as the overturning of a plunging breaker and in our formulation of the method it has been extended so that almost arbitrary configurations of structures and bottom variations can be readily implemented.

The result of the mathematical manipulations is the boundary integral equation for the velocity potential ϕ at a point given by the position vector $\vec{x}$. This equation may be written as

$$\alpha\phi(\vec{x}) = \int_{\Omega}\left[\phi(\vec{x}_0)\,\frac{\partial G(\vec{x},\vec{x}_0)}{\partial n} - G(\vec{x},\vec{x}_0)\,\frac{\partial \phi(\vec{x}_0)}{\partial n}\right] d\omega \qquad (2.1)$$

where $\vec{x}_0$ the position vector for an integration point covering the entire boundary Ω of the flow region considered, and α is a function of the local boundary geometry. See Figure 1 for illustration. The function $G(\vec{x},\vec{x}_0)$ is a Green's function which, strictly speaking, may be chosen in different ways. In all cases, however, analyzed in the fully nonlinear water wave problem or problems with complex boundary geometries G is always the so-called free space Green's function

$$G = \begin{cases} -\ln r & \text{in 2D problems} \\ 1/r & \text{in 3D problems} \end{cases}$$

where r is the distance between points $\vec{x}$ and $\vec{x}_0$.

The solution of the integral equation (2.1) established only the instantaneous flow pattern on the basis of specified boundary conditions. The way in which this flow develops in time is then controlled by the boundary conditions and particularly by the (fully nonlinear) boundary conditions at the free surface, that is, the kinematic condition

$$\frac{d\vec{x}}{dt} = \vec{u} = (\phi_x,\phi_z) \qquad (2.2)$$

and the dynamic condition which is written

$$\frac{D\phi}{Dt} + g\eta - \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \qquad (2.3)$$

where η represents the free surface elevation, $\vec{u}$, the particle velocity, and D/Dt is the time derivative following a particle. Figure 2 shows the boundary curve used in the computations described in the following where we concentrate on wave motion on a slope.

The equation system (2.1), (2.2) and (2.3) can, of course, only be solved numerically, and the solution obtained gives the development in time of a wave or flow pattern specified at the start of the computation, or, as in these computations, generated at the left boundary precisely as by a wave maker in a laboratory facility. At each time step, the integral equation (2.1) is solved (which corresponds to solving Laplace's equation in the flow domain using values of either ϕ or $\partial\phi/\partial n$ specified along the boundary). This solution essentially determines the other (unspecified) quantity (ϕ or $\partial\phi/\partial n$, whichever was not specified as a boundary condition). With both ϕ and $\partial\phi/\partial n$ known along the entire boundary, the two differential equations (2.2) and (2.3) can then be solved bringing the solution one step forward in time. That part of the process both moves the wave surface profile one step in time and, at the same time, establishes

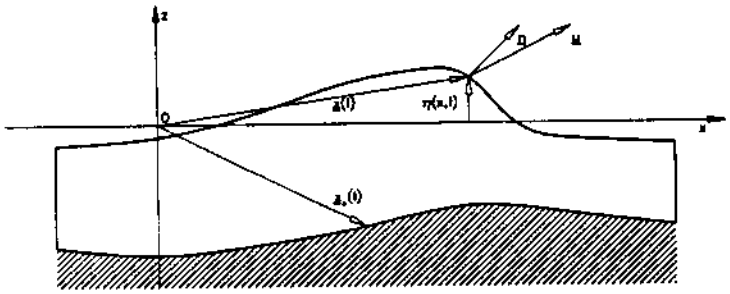

Figure 1. Definition sketch for the BEM method.

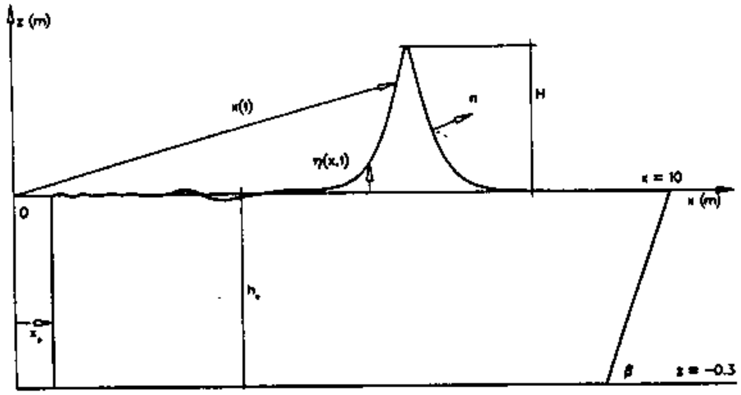

Figure 2. Sketch of the region used in the numerical computations of waves on a steep slope β (x_p denotes the amplitude of the piston wave maker).

the boundary conditions required to solve (2.1) along the new boundary.

3. REVIEW OF BOUNDARY INTEGRAL SOLUTION TECHNIQUES

The present method of solution of the boundary integral problem is innovative in several ways as will be described in the following.

In the most successful applications so far of the method to water wave problems it has been a dominating feature that (2.1) is not solved directly for the physical (x,z) boundary such as shown in Figure 1 or 2. Instead, two dimensionality and the basic properties of potential flow are used to invoke complex function (conformal) mapping techniques which transform the region of solution into simpler boundary forms. Substantial numerical advantages are gained by this technique but also a price is paid. In the first place, the mappings used require that the waves studied are periodic in space. Clearly this is a very severe constraint which excludes most practical problems from being treated.

For a constant depth, the mapping function for a space periodic problem is a relatively simple analytical relationship. LONGUET-HIGGINS and COKELET (1976), VINJE & BREVIK (1981) and DOLD and PEREGRINE (1984) are prominent examples of this technique which has been very successful for the problems considered such as the overturning of breaking waves periodic in space. For any more general boundary geometries or non-periodic waves, however, it is necessary to add a non-trivial step which consists of solving the mapping problem analytically or numerically before the actual solution described in section 2 can start.

The works of COOKER and PEREGRINE (1988) and TELES DA SILVA & PEREGRINE (1989) are examples of this approach.

One of the main features of the method used for the results discussed below is that the integral equation (2.1) is solved directly in the physical space. This choice not only makes it straight forward to deal with situations that are not periodic in space, but it makes it possible to consider almost arbitrary bottom configurations and geometries of structures. A more detailed explanation of this approach and the solution to the problems it represents, has been given in GRILLI *et al.*, (1989).

For any choice of solution technique there are several steps in the process where it is extremely important for the accuracy of the results that sufficient care is exercised in the numerical approximations. One such part is the evaluation of the integrals in (2.1). In most earlier versions, very simple low order approximations have been used for the numerical computation of these integrals which often have influenced adversely the accuracy of the results. A noticeable exception from this is the version developed by DOLD and PEREGRINE (1984) who used high order polynomial approximations for the integrand (but only for space periodic problems).

In some cases the integrals in the equation are computed using shape functions analogous to the finite element method. In this way, high numerical accuracy can be obtained using techniques already developed for that method. The method is then termed a Boundary Element Method (BEM), and that is the version we use here. To the lowest order, this corresponds to a linear approximation within each element of the boundary and the variables defined on the boundary, but the present program has been developed to handle either up to fourth order shape functions for both geometry and variables or quasi-cubic spline approximations.

Another crucial step is the solution of the two partial differential equations (2.2) and (2.3) representing the forward integration in time. In the first applications of the method to nonlinear waves (LONGUET-HIGGINS and COKELET (1976) and others) (2.2) and (2.3) were solved by using an Adams-Bashforth-Moulton (ABM) predictor-corrector method. This procedure yields a high order approximation in time but is not equally accurate in all respects because the $\vec{x}$ derivatives in (2.2) and (2.3) are essentially only determined to the $O(|\Delta\vec{x}|^2)$ or $O(|\Delta\vec{x}|^3)$. The consequence of this is a method which may be of fifth or sixth order in time but only of second order in space. This presumably means that either more points are required to accurately describe the development or the time steps cannot be made as large as the high order accuracy of the ABM method suggests.

It turns out that the method is unstable and develops a so-called sawtooth instability along the free surface. This problem has later been avoided by an alternative approach followed by VINJE and BREVIK (1981), DOLD and PEREGRINE (1984) and others. Their method is replacing the boundary integral equation (2.1) with an equation based on Cauchy's integral theorem. VINJE and BREVIK actually used the same integration technique in time as Longuet-Higgins and Cokelet which suggests that the sawtooth instability is really linked to the solution of the integral equation rather than the forward stepping in time.

More important, however, is the contribution by Dold and Peregrine who developed a high order explicit method for the time integration which is of consistent high order in both space and time. In our computations we have extended that technique to the solution in the physical space. This requires solving the special problems at the boundary corners (Figure 1) which do not exist in the formulation using conformal mapping. In the present version, the method has been implemented to an accuracy of $O(\Delta t)^3$.

4. THE BEM APPLIED TO THE PROBLEM OF WAVES ON STEEP SLOPES

In the following we concentrate on verifying the computational accuracy by comparison with a set of precise measurements and on applying the BEM method to the run-up and reflection of waves on structures with steep front slope such as certain rubble mound breakwaters with relatively smooth fronts, seawalls, and dikes. Quality measurements have been published for the case of a solitary wave on a plane slope by LOSADA *et al.*, (1986) and Figures 3 and 4 show two examples of the comparison between the measured and the computed results. A solitary wave generated by a piston wave maker

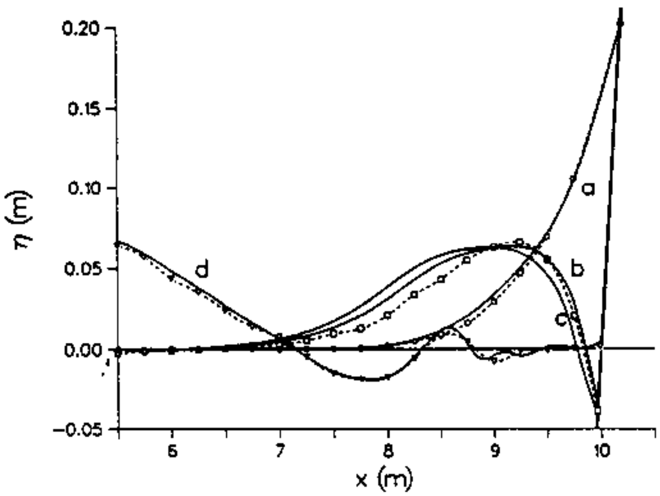

Figure 3a. Comparison between computations (———) and measurements (- - -) by Losada *et al.*, (1986) for a 45° slope. H/h = 0.269. (a) Instant of maximum runup (t = 6.77s in computations). (b) Instant of lowest position of water surface at slope in experiments (t = 7.31s). (c) Instant of lowest surface position at slope in computations (t = 7.38s). (d) t = 9.08s.

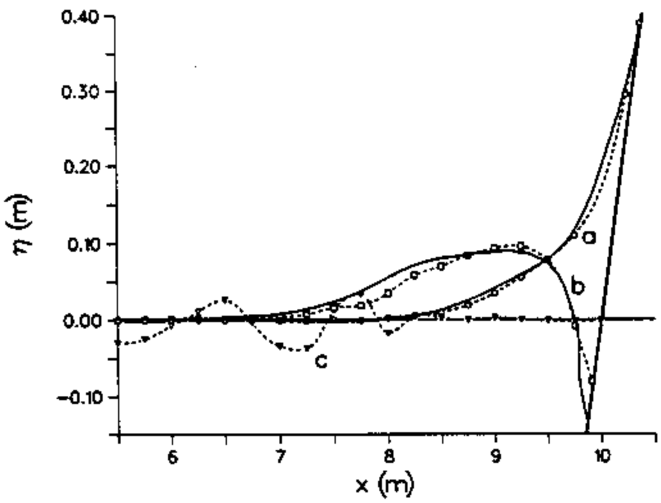

Figure 3b. Same as Figure 3a, 45° slope. H/h = 0.457. (a) Instant of maximum runup (t = 6.14s). (b) Instant of lowest computed surface position at the slope (t = 6.73s). (c) t = 9.18s.

approaches the slope from the left, runs up on the slope and bounces back and the reflected wave moves away to the left. The figures show the surface elevation at several instances of the process. Figure 3 is for a slope of 45° and in Figure 4 the slope is 70°. In both figures two different wave height to water depth ratios have been tested, H/h ~ 0.26–27 (part a) and H/h ~ 0.44–46 (part b) the precise value being slightly different in each case as listed in the figure caption. We see that not only is there a good general agreement, but the computed results, in fact, reproduce even seemingly arbitrary details of the recorded surface variations.

Note that in the very late stage of the run down of the steepest wave on the 45° slope (Fig-

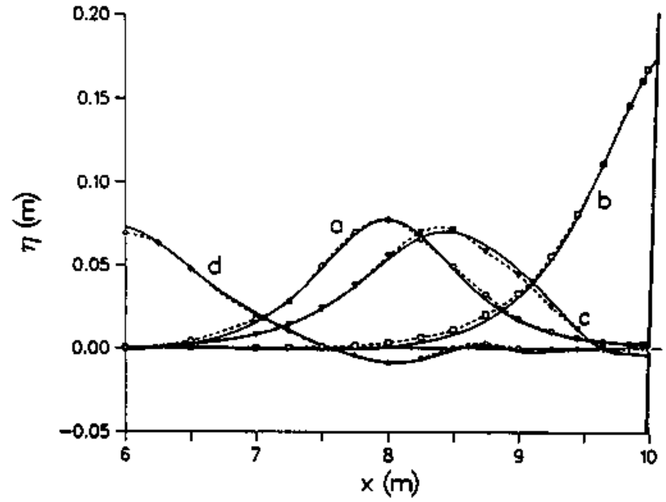

Figure 4a. Same as Figure 3a, 70° slope, H/h = 0.259. (a) Wave profile before maximum runup (t = 5.52s). (b) Instant of maximum runup (t = 6.68s). (c) Instant of lowest surface position at the slope (t = 7.49s). (d) t = 8.76s.

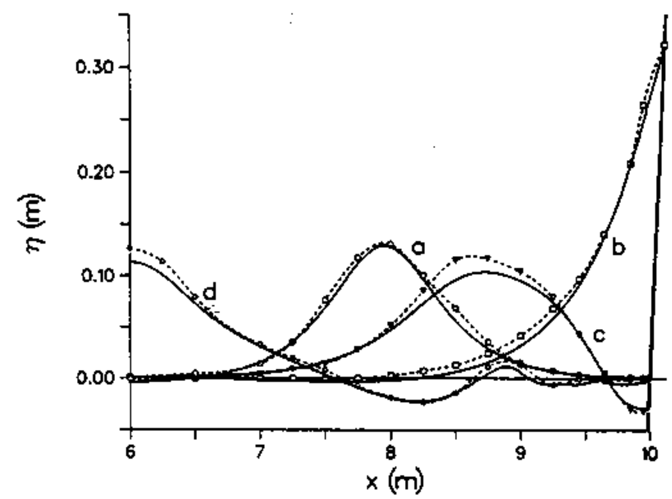

Figure 4b. Same as Figure 3a, 70° slope, H/h = 0.437. (a) Wave profile before maximum runup (t = 4.88s). (b) Instant of maximum runup (t = 6.04s). (c) Instant of lowest position of water surface at the slope (t = 6.67s). (d) t = 7.97s.

ure 3b, curve b), the computation predicts a backward breaking and the computation had to be stopped after the small wave turned over because the computer model cannot describe postbreaking. The experiments also reported such cases of backward breaking. The fact is, however, that the phenomenon is very difficult to predict and describe computationally because it is such a small wave that is only covered by very few computational points when the computer model, like here, is gaged to describe the development in a whole wave flume from generation of the wave at the wave maker until the slope.

The very high degree of agreement between measurements and computations over the entire length of the process studied is, in fact, only possible if the whole flow pattern (velocity field, pressure variations, *etc.*) is similarly accurately modelled. Hence, we can expect the computational model to be able to predict also flow properties such as velocities and pressures that were not measured. Doing so we can get a picture of flow details which would be very difficult and laborious to obtain by direct measurements.

Figures 5 and 6 show an example of such computations for a wave of $H/h \sim 0.46$ on a 30° slope (1:1.73) which is representative for many rubble mound breakwaters. The figures give the development of the internal velocity field during the runup-rundown process. The arrows indicate the magnitude and direction of the velocities at each point at the given instant. From the figures it is possible to follow the process in some detail and get a better feel for how complicated the runup and reflection of a wave on a slope really is.

In addition to the velocity field, Figures 5 and 6 also show the surface position at each instant, and it is striking to see that the time interval between the instant of maximum runup ($t = 6.4s$) and the instant where the water surface almost reaches its lowest position ($t = 6.9s$) is only 0.5s in these experiments where the water depth is 0.3m.

It is also worth noticing that the velocities in the downward part of the motion occurring in Figures 6b-c are somewhat larger than the velocities in the runup phase. If we imagine the slope represents the surface of a breakwater or a seawall, then the situations in Figures 6b-d also represent the time where the internal water table in the porous structure behind the armor layer is at its highest position relative to the external water level which means that the outgoing pressure on the armor blocks is largest at the same time as the drag forces from the down-rush has its numerical maximum (see *e.g.*, BRUUN and JOHANNESSON, 1974 or BARENDS *et al.*, 1983). (The porosity of some structures will influence the flow field somewhat but because of the extremely fast nature of the runup-rundown process, we would expect the pattern shown to be qualitatively the same.) Furthermore, the downslope component of gravity enhances that component of the force. It shows that this phase of the motion is likely to be the most dangerous phase in the runup-rundown sequence and that the blocks in the armor layer failing at this time will be dragged out of the slope and downwards.

5. ALTERNATIVE METHODS FOR WAVES ON STEEP SLOPES

The waves of interest for the design of engineering structures are usually so long that various long wave assumptions can be invoked and this is the basis for most of the alternative theoretical or numerical models used in the literature to analyze the wave interaction with a steep slope. The purely analytical solutions to such models usually can only be obtained for a plane slope whereas the numerical models such as the solution of the nonlinear shallow water equations (and the BEM used in our computations) can actually be used on almost arbitrary slope configurations. In the following two alternative approaches are briefly described and those methods are compared to the BEM by listing advantages and disadvantages of each method. Both are solutions to the nonlinear shallow water equations and one is analytical,

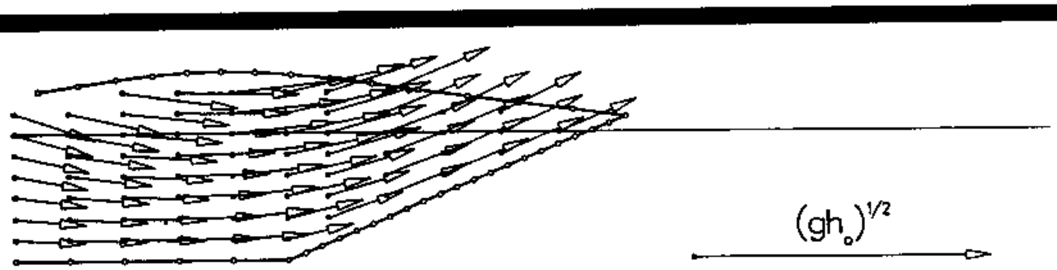

Figure 5a. The internal velocity field during runup of the solitary wave in Figure 3b: 30° slope, $H/h = 0.457$. The times are: (a). 5.50s.

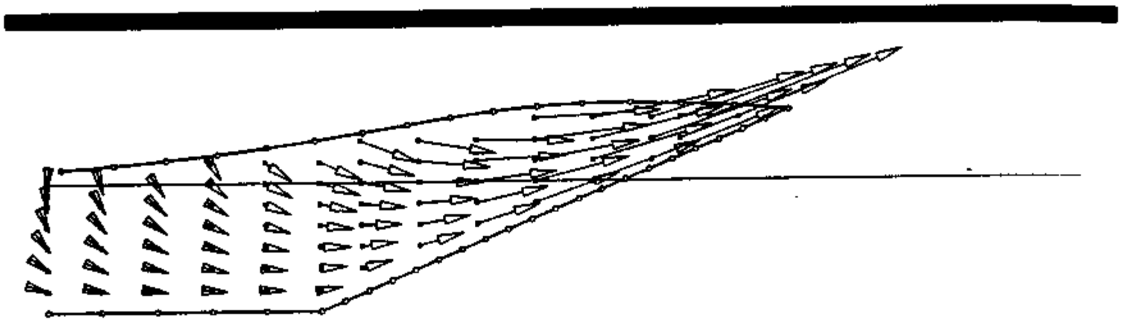

Figure 5b. (b). 5.80s.

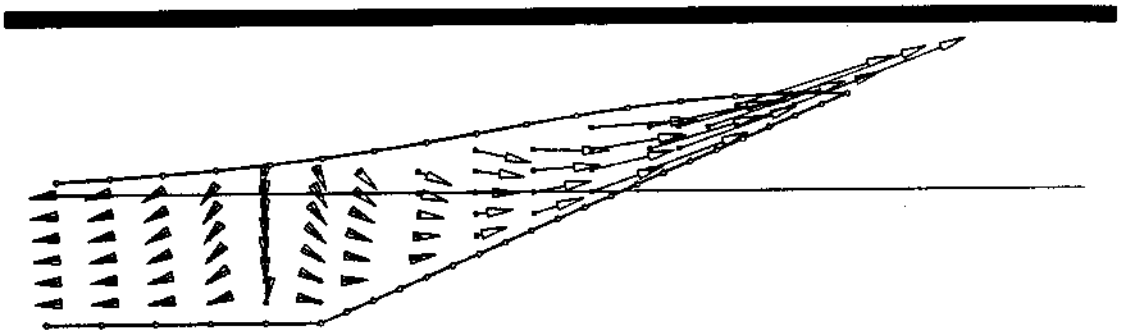

Figure 5c. (c). 5.90s.

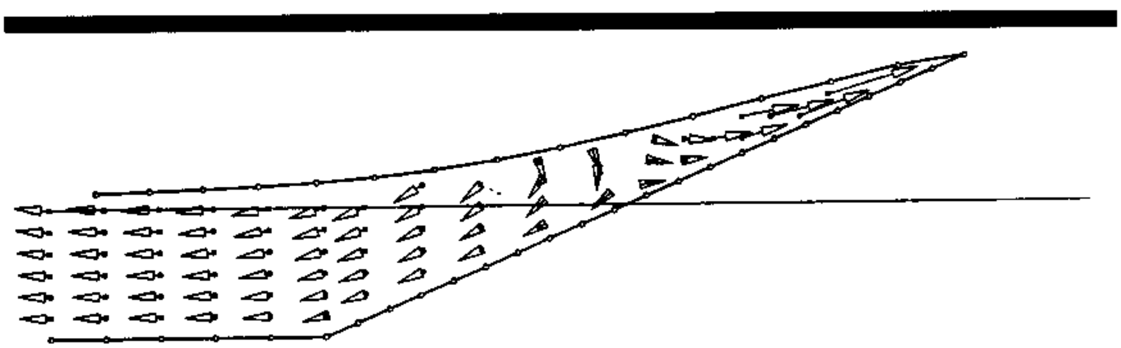

Figure 5d. (d). 6.10s.

one numerical. Since the basic approximations such as hydrostatic pressure and moderate slope utilized in the NSW equations are not required in the BEM, the BEM computations can, to some extent, be used to assess the accuracy of the NSW methods. This particularly applies to slopes of 1:2.5 or steeper which frequently are used in engineering structures, because on such slopes even steep waves do not reach breaking which, as mentioned before, would stop the BEM computations.

Waves on a slope modelled by the NSW equation have been discussed extensively in the literature since CARRIER and GREENSPAN (1958) derived an exact solution to those equations for nonbreaking periodic waves on an infinite plane slope. Numerical solutions of the nonlinear shallow water equations on gentle slopes were first discussed by HIBBERD and PEREGRINE (1979) and the problem has been pursued extensively for both smooth and rough impermeable slopes by KOBAYASHI and coauthors (see, for example, KOBAYASHI *et al.*, (1987), KOBAYASHI and GREENWALD

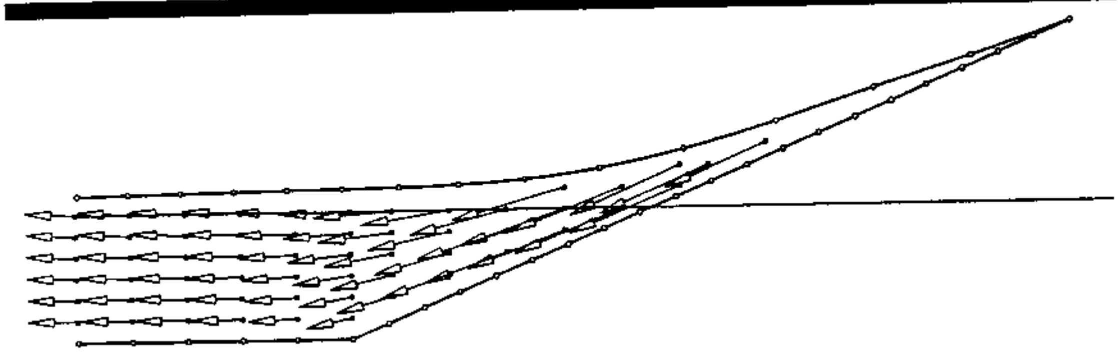

Figure 5e. (e). 6.40s.

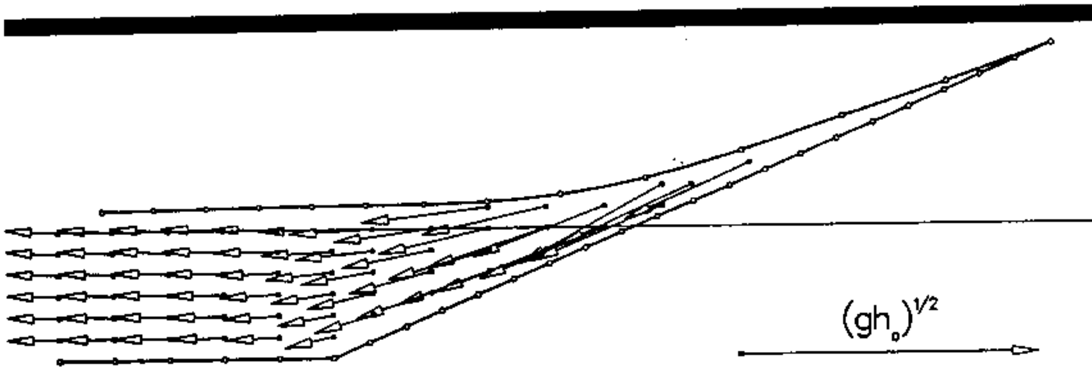

Figure 6a. Same as Figure 5 but for the down rush phase. Times are: (a). 6.45s.

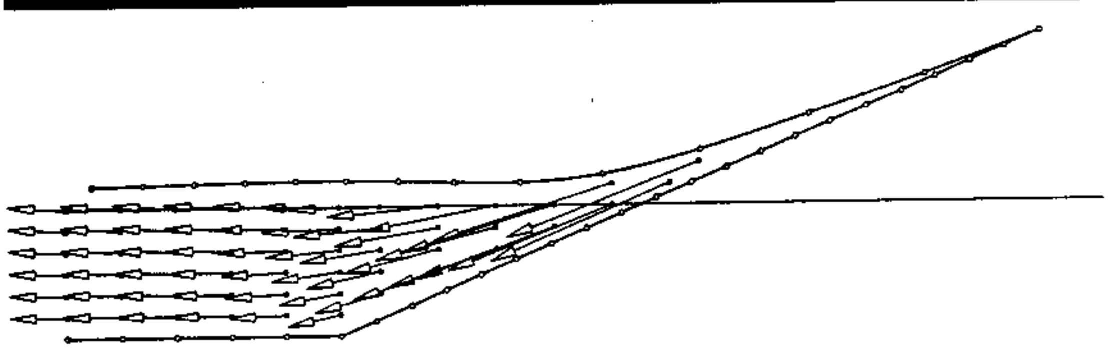

Figure 6b. (b). 6.55s.

(1986), and KOBAYASHI and WATSON (1987)). Also, PEDERSEN and GJEVIK (1983) solved numerically the related Boussinesq equations on a plane slope. An extension of the NSW equation which includes the effect of turbulence due to breaking was derived by SVENDSEN and MADSEN (1984) and solved for a bore on a plane slope.

The NSW equations were originally derived for irrotational waves with amplitudes of the same order of magnitude as the depth but a characteristic horizontal length scale many times the depth. This implies that the dimen-

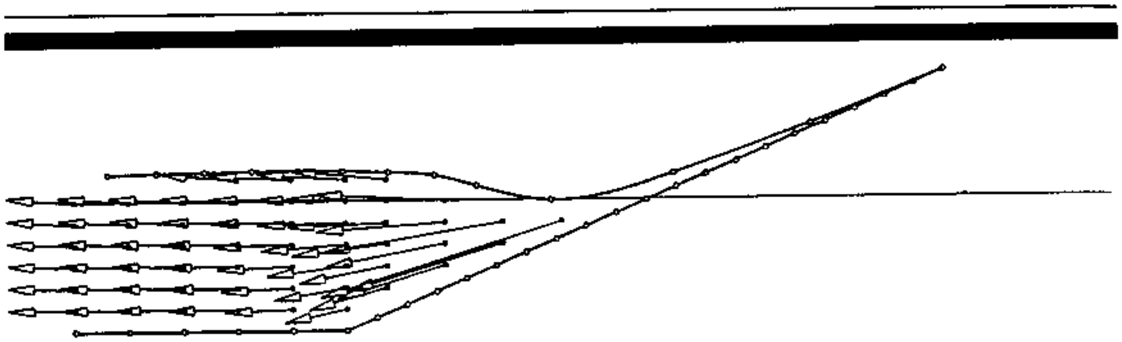

Figure 6c. (c). 6.75s.

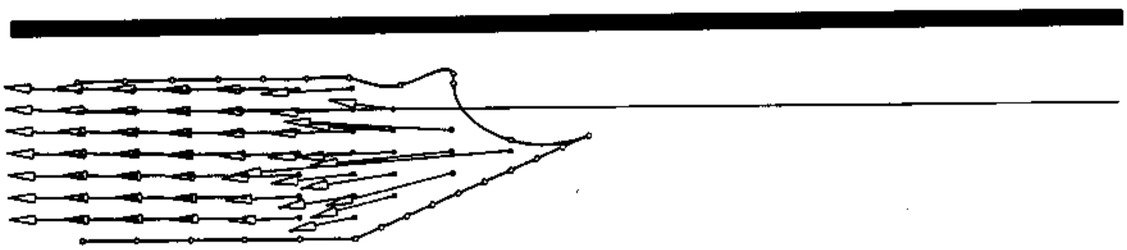

Figure 6d. (d). 6.90s.

sionless ratio $H\lambda^2/h^3$ (the Ursell parameter) between wave height, H, water depth, h, and the horizontal length scale, λ, is assumed large, that is

$$H\lambda^2/h^3 >> 1 \tag{5.1}$$

For a rigorous derivation, hydrostatic pressure need not be assumed but follows as a consequence of (5.1) and, since the flow is potential, bottom friction cannot be incorporated. For a detailed discussion, reference is made to PEREGRINE (1972).

An almost identical set of equations can be derived, however, if hydrostatic pressure is assumed *a priori*. The important advantage of this approach is that bottom friction can be included. Those equations are essentially averaged over depth, and the continuity equation simply becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{5.2}$$

where η is the free surface elevation and Q is defined by

$$Q = \int_{-h_o}^{\eta} u dz \tag{5.3}$$

u being the horizontal velocity at the point and h_o the undisturbed depth. Similarly, the horizontal component of the momentum equation may be written (using $h = h_o + \eta$)

$$\frac{\partial \bar{u}}{\partial t} + \frac{1}{2}\frac{\partial(\alpha \bar{u}^2 h)}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{\tau_b}{\rho h} \tag{5.4}$$

where $\bar{u} = Q/(h_o + \eta)$ is the depth averaged velocity and α is defined as

$$\alpha = \frac{\int_{-h_o}^{\eta} u^2 dz}{\bar{u}^2 h} \tag{5.5}$$

τ_b is the bottom friction. The coefficient $\alpha(>1)$ cannot be determined by the model. For most real velocity profiles, however, α is not much larger than 1 and often it is simply assumed that $\alpha = 1$ which implies the horizontal velocity is uniform over depth, and this is the version of the NSW equations used in the references quoted above.

Though valid on uneven bottoms, the validity of the NSW approximation requires that $\tan^2\beta << 1$ (HIBBERD and PEREGRINE, 1979 and KOBAYASHI *et al.*, 1987). How large values of $\tan^2\beta$ can be allowed cannot be derived from the model. Due to the many other advantages of the NSW model, it is desirable to analyze its range of application even on steeper slopes and it is the purpose of the following comparison between the BEM and the NSW methods to analyze how seriously the solution deviates when applied to slopes typical for rubble mound breakwaters. We can do this by looking at how well the assumption of depth uniform velocity for cases without friction and the assumption of hydrostatic pressure are actually satisfied in the exact computations and also by comparing other results such as the predictions of the maximum run-up.

Figure 7 shows horizontal velocity vectors derived from the computational solution of the potential problem for $\beta = 30°$ ($\tan^2\beta = 0.333$) discussed earlier. If we assume that the NSW solution would predict exactly the same surface elevations in space and time as the BEM (this problem is not investigated here), then the frictionless NSW equations would predict the (depth constant) mean U of each of the velocity profiles shown in the figure. We see from Figure 7 that in most of the situations shown, the velocity variation over depth varies up to $\pm(25-35)\%$ from the depth averaged value, with the smallest values near the bottom. Therefore, depending on the problem, the depth averaged velocity may or may not represent a reasonable approximation to the real situation.

Strictly speaking, the NSW approximation only deals with horizontal velocity components, because in the long waves implied by (5.1) the vertical velocities are assumed negligible. On the other hand, the vertical velocities actually occurring can be calculated from the local continuity equation which hence is not directly invalidated by the NSW equations. This is particularly important when the bottom slope becomes as steep as it is on many engineering structures because the sloping bottom means that if there are horizontal velocities then there are also vertical velocities which are at least ~ $\bar{u}\tan\beta$ ($=0.577\,\bar{u}$ in the example $\beta = 30°$). Such vertical velocities are associated with vertical accelerations which require deviations from hydrostatic pressure. Hence we must expect that as the slope becomes steeper that fundamental assumptions may not hold.

Figure 8 shows the pressure along the surface of the slope, the position being measured by the vertical coordinate z. The ordinate on the figure is the actual computed pressure p_T divided by the hydrostatic pressure, $p_H = \rho g(h_o + \eta)$ that would be assumed at that point and instant by the nonlinear shallow water theory. Thus, $p_T/p_H = 1$ represents the hydrostatic hypothesis of the NSW equation exactly satisfied. The pressure variation is shown for each of the instants for which the velocity field is given in Figures 5 and 6.

In essence, the deviation from hydrostatic pressure indicates the degree to which the horizontal accelerations driving the flow deviate from the assumed $g\partial\eta/\partial x$. Thus, the rather substantial deviations in pressure suggests that the temporal development of the surface profile that can be obtained from the NSW model may deviate similarly from the exact computations. However, this hypothesis needs to be studied further.

6. COMPARISON WITH AN ANALYTICAL SOLUTION FOR RUNUP HEIGHT

On a plane slope going to infinite depth, the NSW equations have been solved analytically for frictionless flow by CARRIER and GREENSPAN (1958) and this solution has recently been extended by SYNOLAKIS (1987) to the more interesting case of a plane slope in a region with an otherwise finite constant water depth. Results for a solitary wave are available and can be compared with our computations

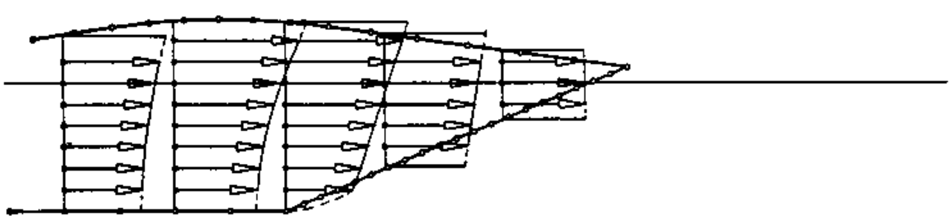

Figure 7a. Profiles of horizontal velocity components for some of the situations shown in Figures 5 and 6. Times are: (a). 5.50s.

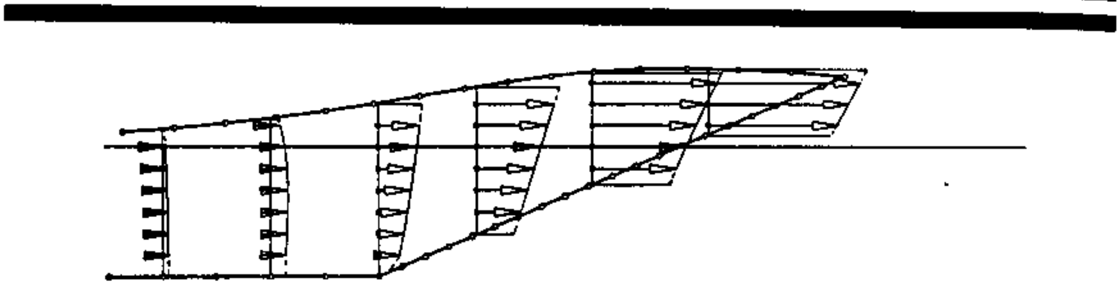

Figure 7b. (b). 5.80s.

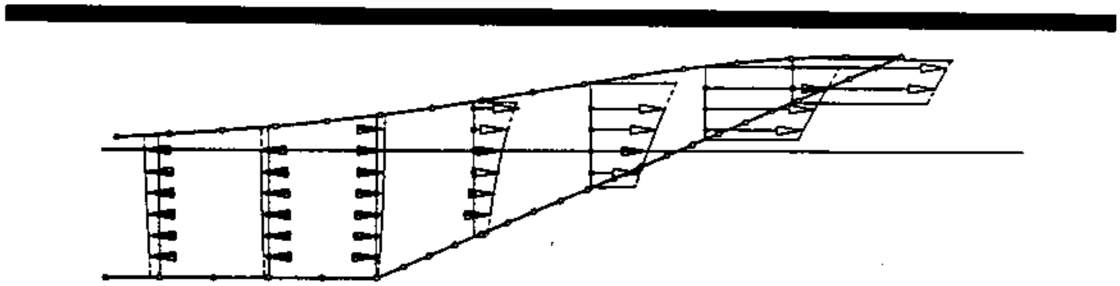

Figure 7c. (c). 5.90s.

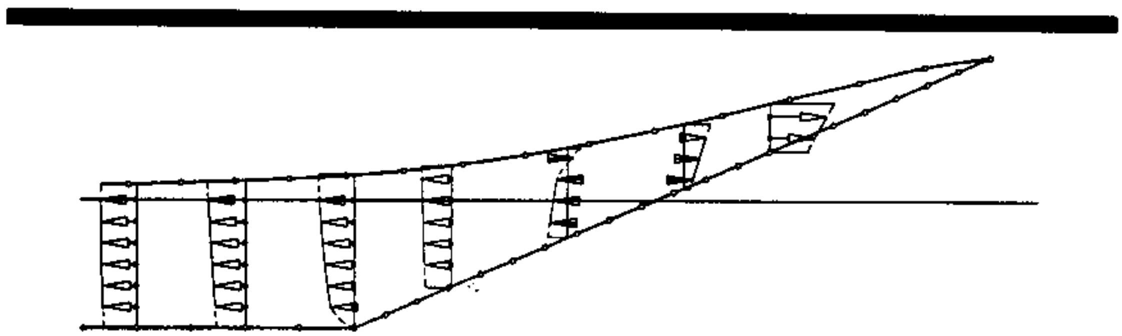

Figure 7d. (d). 6.10s.

and also with the measurements of LOSADA *et al.*, (1986).

Results for the maximum runup height, R, are shown in Table 1 for slope angles ranging from 2.88° (approximately 1:20) to 70°. The second column in the table indicates the height of the incident waves relative to the water depth, h_o, in front of the slope. R_e in the third column is then the experimental values obtained by LOSADA *et al.*, (1986) (70° and 45°), by HALL and WATTS (1953) (15°), and by SYNOLAKIS (1987) (2.88°). Among the computed results R_u is the runup determined by the BEM method, ("_____" indicates breaking before maximum runup).

Synolakis gives two estimates of the runup: an approximate value

$$R_s/h_0 = 2.831(\cot\beta)^{1/2}(H/h_0)^{5/4} \qquad (6.1)$$

where β is the slope angle. This expression is only valid on relatively gentle slopes satisfying the requirement

$$A = 0.288\,\frac{\tan\beta}{(H/h_0)^{1/2}} << 1 \qquad (6.2)$$

where H is the incident wave height and h_0 the depth in front of the slope.

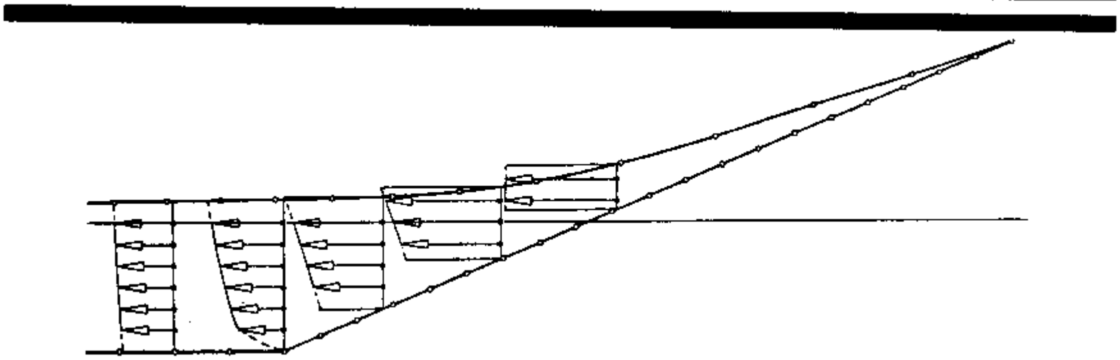

Figure 7e. (e). 6.40s.

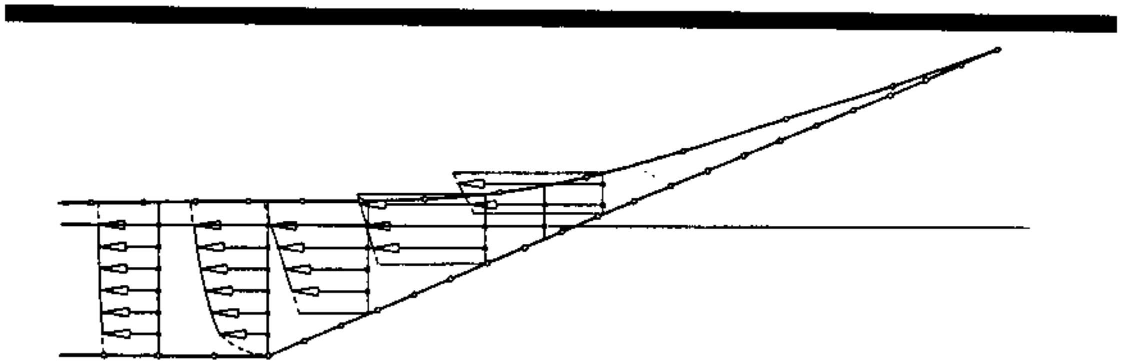

Figure 7f. (f). 6.45s.

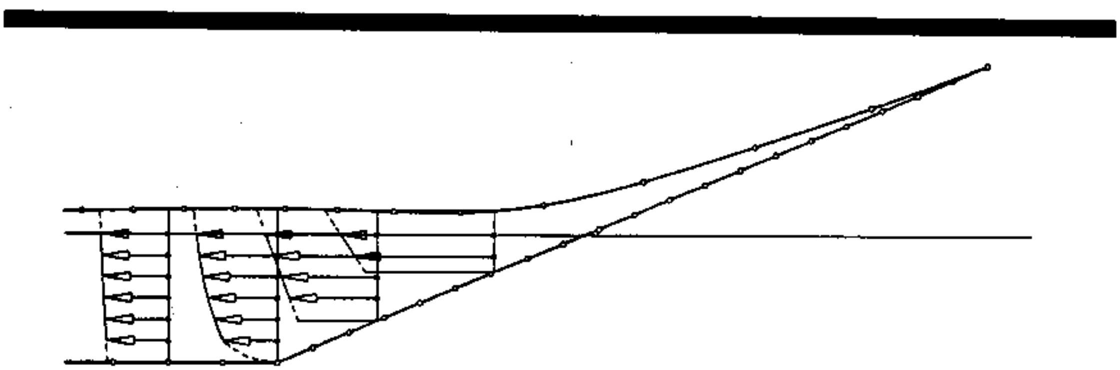

Figure 7g. (g). 6.55s.

The other expression is an integral relation for R. The results from this equation listed in the table as R_I are from SYNOLAKIS (1989) and are only available for the steep slopes where the approximate expression (6.1) does not apply.

It is fairly clear from the table that within their range of applicability, all three methods yield results that are of the same order of magnitude as the measured values. It is also evident, however, that the methods are not equally accurate.

As far as (6.1) is concerned, it appears that the limit which (6.2) specifies for the validity of that approximate equation is very real. The parameter A is listed in Table 1. Evidently, for

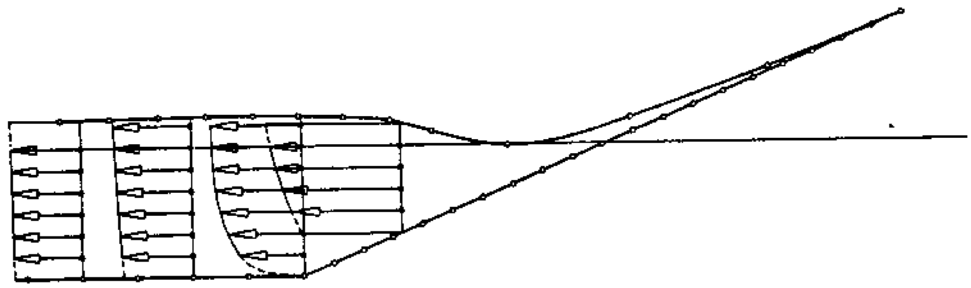

Figure 7h. (h). 6.75s.

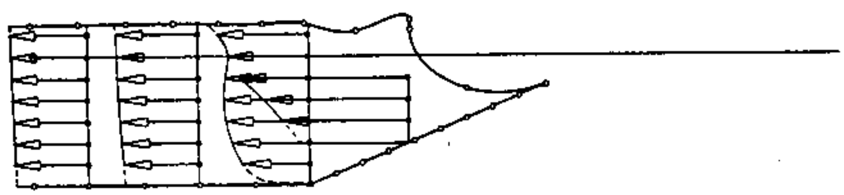

Figure 7i. (i). 6.90s.

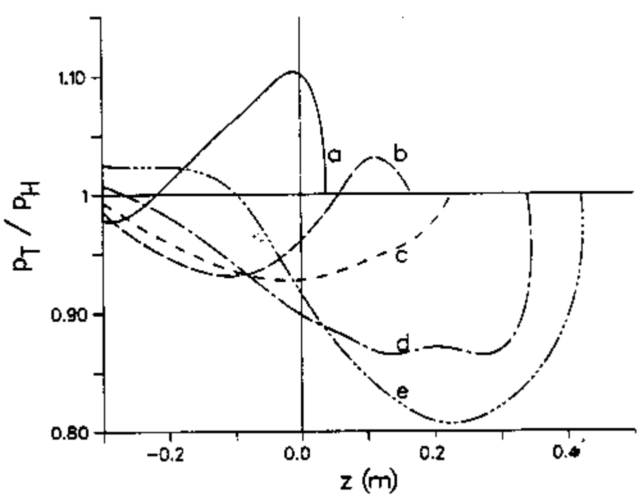

Figure 8a. Pressure variation at points along the slope for the same cases as in Figure 5.

the steep slopes of 45° and 70°, A is not <<1 and, consequently, the R_s/h_o from (6.1) cannot be expected to predict the runup. The values turn out to be only about half the measured values. For 15° and 2.88° slopes (6.2) is satisfied and nevertheless the predictions of (6.1) are only reasonably accurate for the small wave (H/h_o = 0.1) on a 15° slope (error + 9.6%, A = 0.244). Although A is smaller (0.173) for the steeper wave (H/h_o = 0.2) on that slope the predictions of the runup is actually less accurate (error +22.2%) and for a 2.88° slope, the errors are +15.6% and +44.9%, respectively, although A ~ 0.1 (<< 1) which means that (6.2)

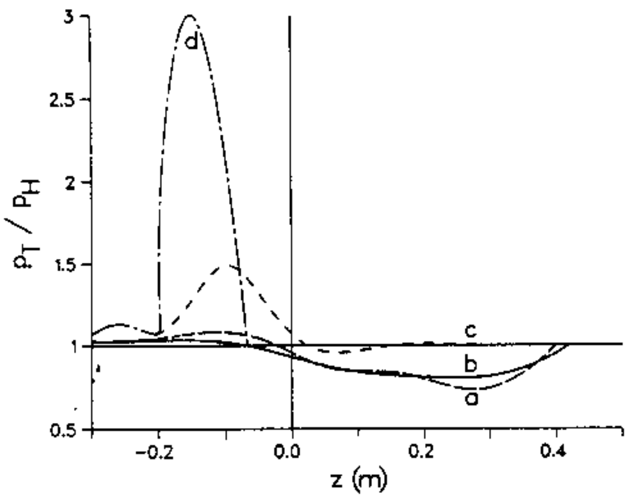

Figure 8b. Pressure variation at points along the slope for the same cases as in Figure 6.

Table 1. *Comparison of numerical and analytical results with experimental results*

	Experimental results			Numerical and analytical results			
β	H/h_o	R_e/h_o	Exper. by	R_u/h_o BEM	R_e/h_o (6.1)	R_l/h_o (Int eq)	A (6.2)
70°	0.259	0.558	LOSADA, et al.	0.580	(0.316)	0.524	1.560
70°	0.437	1.071	(1986)	1.062	(0.607)	1.142	1.200
45°	0.269	0.672		0.674	(0.546)	0.644	0.560
45°	0.457	1.294		1.257	(1.064)	1.194	0.430
15°	0.100	0.281	HALL & WATTS	0.310	0.308		0.244
15°	0.200	0.599	(1953)	0.654	0.732		0.173
2.88°	0.019	0.077	SYNOLAKIS	0.081	0.089		0.105
2.88°	0.040	0.156	(1987)	—	0.226		0.072

is satisfied. The data in SYNOLAKIS (1987) actually suggest that the last wave may be breaking (as our computations predict) which would explain the large discrepancy. As will be argued later, part of that error is likely to be due to the fact that the prediction (6.1) does not include the effect of friction which does become important, in particular in laboratory flumes when the slope is as small as 2.88°.

It is also worth noting that the accuracy of (6.1) in all cases is less for the steeper waves on each slope although the form of (6.2) yields smaller A-values for such waves and, hence, implicitly suggests that (6.1) should be more accurate the steeper the wave. So that deduction clearly is not confirmed by the results.

The runup R_l predicted by the integral equation derived by SYNOLAKIS are reasonably accurate for steep slopes, errors ranging from −4.2% (45°, H/h_o = 0.269) to −7.7% (45°, H/h_o = 0.457). This, however, is probably the best accuracy one can expect because although the integral equation is claimed to be exact, this only means an exact result for first-order solitary waves. The largest errors occur for the steepest waves where the first-order theory is least accurate.

In comparison, the BEM method is seen to work very well, particularly on the steep slopes of 45° and 70° where the error on the runup is +3.9%, +0.8%, +0.3% and +2.9%, respectively for the four cases in Table 1. In fact, this is probably within the accuracy of the well-controlled measurements themselves, so that those figures cannot really be considered a measure of "error" relative to a flawless experimental result.

On the 15° slope (1:3.73), the BEM predicts too large runup figures for both waves (+10.3% and 9.2%, respectively).[1] Since the accuracy of the computations is not influenced by the slope angle, β, (as long as the waves do not plunge), the only explanation for the deviations is that friction in the experiments causes the difference. This difference, however, would then be smaller in large scale nature because the friction is larger in the laboratory experiments both due to smaller Reynolds number and due to side wall effects. This also throws some more advantageous light on the accuracy of the approximate formula (6.1) as far as the less steep waves are concerned.

The results for the mildest slope ($\beta = 2.88°$) really confirm this. The BEM could only predict runup for the smaller of the two waves tested because the wave with height ratio 0.04 broke before the runup point was reached. The result available, however, agrees with the measurements to within +5.2% (against +15.6% for (6.1)) which confirms that for the relatively small amplitude wave (height ratio 0.019), the experimental friction is of relatively smaller importance. Also, the experiments of SYNOLAKIS are believed to be more accurate than the much older measurements by HALL and WATTS (1953).

7. DISCUSSION OF RESULTS AND THEIR LIMITATIONS

It has been demonstrated here that the BEM method which essentially solves the exact equations for flow without friction is a highly accurate method for prediction of nonlinear wave motion. This can be done in regions with fairly arbitrary bottom topographies and boundary geometries. It has been demonstrated numerous times that the method is also capable of predicting the overturning of the wave as in a plunging breaker and the version developed for the present computations has that capability as well. Prediction of the motion of the falling jet itself requires far more computing accuracy than is necessary for the motion of nonlinear waves in general, but essentially this is a matter of caring for some computational details within the framework of the model.

However, once the plunging jet hits the water in front of the crest and turbulence starts to develop, the basic assumptions for the model breakdown. Although it may be possible to develop the model to carry the computations somewhat further than the point where the jet plunges, this is the time where it is necessary in the present version to stop the computations. Hence, the real limit of applicability of the BEM models today is that they cannot handle breaking anywhere in the computational region.

The exact analytical solution of the NSW equations developed by SYNOLAKIS (1987) is limited in generality in the first place by only considering a solitary wave as input and by only representing a solution for a plane slope. Secondly, it is limited in accuracy by only considering a first-order solitary wave as input. The solution does have features that Carrier and Greenspan have termed wave breaking, simply because the solution for the surface for some parameter values develops a vertical tangent as would be the case in real wave breaking. This type of "breaking" has been investigated quite extensively in the literature from the mathematical point of view but has, to the knowledge of the authors, never been compared in details with experimental or accurate computational results as those of the BEM. We are suspicious though of the relevance of this breaking because the assumptions underlying the NSW-equations on which it is based break down long before the tangent to the water surface becomes vertical. The importance of this argument is further enhanced by the fact that the Serre equations (SERRE, 1953) do not show a behavior similar to the NSW equations as demonstrated, *e.g.*, by the numerical work by SEABRA-SANTOS *et al.*, (1987). Those equations are a higher order approximation to the long wave problem that includes the effects of vertical accelerations while keeping the Ursell parameter $>>1$.

It would be expected that a numerical solution of the NSW-equations must suffer from all the limitations imbedded in the equations themselves. However, the solution technique used by HIBBERD and PEREGRINE (1979) and KOBAYASHI *et al.*, (1987) is based on a dissipative LAX-WENDROFF scheme which has features not imbedded in the original NSW equations. LAX and WENDROFF (1960), suggested to add artificial dissipation to the

1. Those deviations are of the same order of magnitude as for the approximate method although the accuracy is independent of the wave height because the BEM is equally accurate for steep waves.

numerical scheme they developed themselves by rigorous Taylor expansions. For details see RICHTMYER and MORTON (1967). The dissipation is designed so that mass and momentum is conserved and the dissipation remains small for waves with limited surface curvature. On the other hand, since the NSW equations do not have solutions of constant form even on a horizontal bottom, any wave propagated by those equations will steepen until the front slope of the water surface becomes vertical, just as in the analytical solution on a slope by Carrier and Greenspan. In the numerical solution using the dissipative Lax-Wendroff scheme, however, the dissipative terms become large just where the surface slope becomes "large". This has several effects. First, it stops further steepening of the profile which now attains a constant shape with a very steep (but not vertical) front slope, very much like in a real wave that propagates while it continues to break (HIBBERD, 1977). Second, it may be inferred from the results of SVENDSEN *et al.*, (1978) that if the shape of the wave has become stagnant, the energy dissipation will equal that of a hydraulic jump with a height equal to the wave height. SVENDSEN (1984) has shown that such a dissipation will render a good prediction of the wave height development on a gentle slope once the wave has developed a bore like shape, although it does not predict the transition of the breaker shortly after breaking, in particular, not for plunging breakers. Hence, on a gentle slope where the NSW equations are normally used (see section 5) the dissipative Lax-Wendroff scheme applied to the NSW equations supplies a solution which, in many respects, closely models the actual surf zone wave. The two major limitations are that the point of breaking is determined in a non-physical way as the time it takes the wave to steepen, which for a wave of a given height to depth ratio is a certain distance from the starting point of the computations. If, in other words, we move the starting point of the computations then we move the predicted breaking point similarly. The second limitation of the realism of the predicted wave is that the steepness of the stabilized front is determined by the discretization step Δx. HIBBERD and PEREGRINE (1979) found that the transition from trough to crest would occur over 4–5 times Δx. Hence, changing Δx changes the front shape of the wave.

On a steep slope it is less clear how to interpret the results because, although the front steepens and the energy dissipation becomes large, the dispersive mechanisms which are important for that process are missing in the NSW equations. Also the front shape can hardly get time to reach the stable form which was the basis for the above mentioned resemblance with the energy dissipation in a hydraulic jump. Finally, we have seen that the deviations from hydrostatic pressure may be important. Hence, a closer investigation of those details would be necessary before relying on the NSW equations for wave motion on steep slopes.

8. CONCLUDING REMARKS

As we have seen, none of the methods available are capable of describing all aspects of the wave motion on steep slopes.

The BEM is accurate and gives the most details but cannot be pursued beyond breaking, which limits its application to the steeper slopes. Also in its present form, bottom friction and structural porosity are not included.

The analytical solution of the NSW equations has severe limitations in being limited to simple wave shapes (presently solitary waves) and plane slopes.

The most frequently used numerical scheme for the NSW equations has been well verified on gentler slopes (less than 20°) and for those cases renders a robust model that includes bottom friction and can carry the computations beyond the breaker point. The equations, however, cannot be expected to work well on slopes of 30° or more. Also some caution would need to be exercised as to the interpretation of the results for how the waves develop towards breaking.

Hence it is concluded that on steep slopes the BEM method is likely to give the most accurate results although it presently does not include bottom friction and cannot be carried beyond the instant where the jet of a breaking wave hits the surface in front of the crest.

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LITERATURE CITED

BARENDS, F.B.J.; KOGEL, H.V.D.; UIJTTE WAAL, F.J., and HAGENAAR, J., 1983. West breakwater—Sines: Dynamical-geotechnical stability of breakwaters. *Proceedings ASCE Conference Design, Maintenance and Performance of Coastal Structures* (Arlington, Virginia).

BRUUN, P., 1976. *Port Engineering*. Houston, TX: Gulf Publishing Company, 586p.

BRUUN, P. and JOHANNESSON, P., 1974. A critical review of the hydraulics of rubble mound structures. Div. Port and Ocean Engineering, University of Trondheim, Norway, *Institute Report R3*.

CARRIER, G.F. and GREENSPAN, H.P., 1958. Water waves of finite amplitude on a sloping beach. *Journal Fluid Mechanics, 4*(1), 97–110.

COOKER, M. and PEREGRINE, D.H., 1988. Solitary waves passing over submerged breakwaters. *Proceedings 21st International Conference Coastal Engineering* (Malaga, Spain) pp. 624–631.

DOLD, J.W. and PEREGRINE, D.H., 1984. Steep unsteady water waves: An efficient computational scheme. *Proceedings 19th International Conference Coastal Engineering* (Houston) pp. 955–967.

GRILLI, S.T.; SKOURUP, J., and SVENDSEN, I.A., 1989. An efficient boundary element method for nonlinear water waves. *Engineering Analysis with Boundary Elements*, 6(2), 97–107.

HALL, J.V. and WATTS, J.W., 1953. Laboratory investigation of the vertical rise of solitary waves on impermeable slopes. Beach Erosion Board, U.S. Army Corps of Engineers, *Technical Memo No. 33*, 14 pp.

HIBBERD, S., 1977. Surf and run-up. Ph.D. dissertation, School of Mathematics, University of Bristol.

HIBBERD, S. and PEREGRINE, D.H., 1979. Surf and run-up on a beach: a uniform bore. *Journal Fluid Mechanics, 95*, 323–345.

KOBAYASHI, N. and GREENWALD, J.H., 1986. Prediction of wave run-up and riprap stability. *Proceedings 20th International Conference Coastal Engineering*, ASCE, pp. 1958–1971.

KOBAYASHI, N. and WATSON, K.D., 1987. Wave reflection and runup on smooth slopes. *Proceedings Coastal Hydrodynamics*, ASCE, pp. 548–563.

KOBAYASHI, N.; OTTA, A.K., and ROY, I., 1987. Wave reflection and runup on rough slopes. *ASCE, Journal Waterway, Port, Coast, and Ocean Engineering, 113*(3), 282–298.

LAX, P. and WENDROFF, B., 1960. Systems of conservation laws. *Comm. Pure Appl. Math. 13*, 217–237.

LONGUET-HIGGINS, M.S. and COKELET, E.D., 1976. The deformation of steep surface waves on water–I. A numerical method of computation. *Proceedings Royal Society London A. 350*, 1–26.

LOSADA, M.A.; VIDAL, C., and NUNEZ, J., 1986. Sobre El Comportamiento de Ondas Propagadose por Perfiles de Playa en Barra y Diques Sumergidos. Dirección General de Puertos y Coastas, Programa de Clima Maritimo, Universidad de Cantábria. *Publicacion No. 16*.

PEDERSEN, G. and GJEVIK, B., 1983. Run-up of solitary waves. *Journal Fluid Mechanics, 135*, 283–299.

PEREGRINE, D.H., 1972. Equations for water waves and the approximations behind them. In: R.E. Meyer, ed., *Waves and Beaches*, New York: Academic Press.

RICHTMYER, R.D. and MORTON, K.W., 1967. Difference methods for initial value problems. New York Interscience.

SEABRA-SANTOS, F.J.; RENOUARD, D.P., and TEMPERVILLE, A.M., 1987. Numerical and experimental study of the transformation of a solitary wave over a shelf or isolated obstacle. *Journal Fluid Mechanics, 176*, 117–134.

SERRE, F., 1953. Contribution a l'étude des écoulements permanents et variables dans les canaux. *La Houille Blanche*, pp. 374–388.

SVENDSEN, I.A., 1984. Wave heights and set-up in a surf zone. *Coastal Engineering*, 8(4), 303–329.

SVENDSEN, I.A. and MADSEN, P.A., 1984. A turbulent bore on a beach. *Journal Fluid Mechanics, 148*, 73–96.

SVENDSEN, I.A.; MADSEN, P.A., and BUHR HANSEN, J., 1978. Wave characteristics in the surf zone. *Proceedings, 16th International Conference Coastal Engineering*, (Hamburg) ASCE, pp. 520–539.

SYNOLAKIS, C.E., 1987. The runup of solitary waves. *Journal Fluid Mechanics, 185*, 523–545.

SYNOLAKIS, C.E., 1989. *Private communication.*

TELES DA SILVA, A.F. and PEREGRINE, D.H., 1989. The nonlinear interaction between a free surface potential flow and a submerged cylinder. *Proceedings 4th International Workshop on Water Waves and Floating Bodies* (Oystese, Norway), pp. 51–54. Department of Mathematics, University of Oslo.

VINJE, T. and BREVIK, P., 1981. Numerical simulation of breaking waves. *Advances Water Resources*, 4, 77–82.