Multimode damage tracking and failure prognosis in electromechanical system

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ABSTRACT

In this paper a modification to a general-purpose machinery diagnostic/prognostic algorithm that can handle two or more simultaneously occurring failure processes is described. The method is based on a theory that views damage as occurring in a hierarchical dynamical system where slowly evolving, hidden failure processes are causing nonstationarity in a fast, directly observable system. The damage variable tracking is based on statistics calculated using data-based local linear models constructed in the reconstructed phase space of the fast system. These statistics are designed to measure a local change in the fast systems flow caused by the slow-time failure processes. The method is applied to a mathematical model of an experimental electromechanical system consisting of a beam vibrating in a potential field created by two electromagnets. Two failure modes are introduced through discharging batteries supplying power to these electromagnets. Open circuit terminal voltage of these batteries is a two-dimensional damage variable. Using computer simulations, it is demonstrated both analytically and experimentally that the proposed method can accurately track both damage variables using only a displacement measurements from the vibrating beam. The accurate estimates of remaining time to failure for each battery are given well ahead of actual breakdowns.

Keywords: Dynamical systems, diagnosis, prognosis, multimode damage tracking and failure prediction

1. INTRODUCTION

Damage diagnosis and failure prognosis in engineered systems is one of the important current problems of applied engineering research. The progress in developing technologies for system health monitoring and condition based maintenance is hampered by the hidden nature of damage. The damage physics is in its developmental stage: in many cases actual damage mechanisms are not known, or are hard to model mathematically for diagnostic and prognostic purposes. The extent of damage in particular systems can usually be assessed by various means, after removing machinery from operation or destructive testing. This results in serious productivity and capital losses. The implementation of on-line, nondestructive damage diagnosis and prognosis technology that utilizes readily available measurements can bring substantial savings to the industry without compromising environmental and human safety.

A comprehensive review of damage identification and system health monitoring methods is presented in Ref. 1. A more recent review article of methods relating to health monitoring of composite structures is presented in Ref. 2. In addition to these papers, the following paragraphs give a brief representative overview of the main developments in the field.

Most of the earlier damage identification work focused on diagnosis, and in particular, damage detection. Several strategies exist for tackling this type of problem.3, 4 One data-based, or heuristic, approach is to look for changes due to the damage accumulation in time or frequency domain statistics,5–8 or in statistics that have both time and frequency information.9–12 For nonlinear systems exhibiting chaotic response it is customary to use estimates of long-time chaotic invariant measures, such as a correlation dimension.13–15 Other advanced techniques use expert systems or fuzzy logic16–18 and genetic algorithms.19 The main advantages of such methods are simplicity of implementation and that they often work very well. Most heuristic methods serve as purely damage detection methods, i.e. no damage state assessment is provided. Even when the severity of damage can be estimated,20 it is usually very hard to establish a direct one-to-one connection between

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the damage state and the change in the heuristic statistic or feature vector. There is no theoretical basis for predicting a priori, without the benefit of a good model or experiment, whether a certain feature vector will work well, or not, for a particular system.

Another model-based approach addresses some of the shortcomings of the purely statistical approach, typically at the expense of more difficult development and implementation.\textsuperscript{21–23} In cases where the system’s analytical model is available it is usually possible to establish a functional connection between the drifting parameters and a particular feature vector.\textsuperscript{24} However, due to the general difficulty of developing physics-based mathematical models for many systems, such analytical models are not usually available.

The lack of analytical models is customarily addressed by developing finite element or data-based models. Some examples for linear systems are autoregressive models\textsuperscript{25} or frequency response functions.\textsuperscript{26, 27} Nonlinear systems are usually modeled using neural networks.\textsuperscript{28–31} Other successful approaches are based on some type of hybrid method. For example, extensive attention is allotted to the use of mode shapes, or their curvatures, for damage detection and identification.\textsuperscript{32–34} However, in many cases these methods are application dependent, and the main advantage of a model-based approach, to correlate the changes in a feature vector with the changes in a system’s physical parameters, is lost.

The failure prognostics problem, by itself, is still in the developmental stages. Currently available prognostic methods can be divided into methods based on deterministic\textsuperscript{35} and probabilistic or stochastic\textsuperscript{36, 37} modeling of fault or damage propagation. The given methods are still application dependent, since they are closely tied to a particular damage detection problem. In addition, these methods cannot be considered general or comprehensive solutions, since their applicability is contingent upon successful damage state assessment which is provided by some suitable damage detection method.

In previous work\textsuperscript{38, 39} an experimental method for tracking and predicting one-dimensional damage process in dynamical systems was presented. The potential of the method to address above mentioned problems was clearly demonstrated. This method was based on viewing damage as a slow process evolving in a hierarchical dynamical system consisting of a “fast-time,” directly observable subsystem coupled to a “slow-time,” hidden or damage subsystem. Here, a modification to the experimental damage tracking algorithm that can handle multi-dimensional damage processes is presented. Consider a systems of the form:

\begin{align}
\dot{x} &= f(x, \mu(\phi), t), \\
\dot{\phi} &= \epsilon g(x, \phi, t),
\end{align}

\textsuperscript{(1a, b)}

where $x \in \mathbb{R}^n$ is the directly observable fast dynamic variable of subsystem Eq. (1a); $\phi \in \mathbb{R}^m$ is the slow dynamic variable (“hidden” damage state) of subsystem Eq. (1b); the parameter vector $\mu \in \mathbb{R}^p$ is a function of $\phi$; $t$ is time; and the rate constant $0 < \epsilon \ll 1$ defines the time scale separation between the fast dynamics and the slow drift. Note that $\epsilon = 0$ corresponds to the constant parameter case (i.e., Eq. (1a) is stationary: $\mu = \mu(\phi_0) \equiv \mu_0$).

In the experimental procedure, we assume that a scalar, $C^2$ function of fast variable $x$ is measured and recorded. The phase space of Eq. (1a) is reconstructed, via delay coordinate embedding, using the measured scalar time series recorded over intermediate time scales (times which are long compared to the fast dynamics but short compared to the drift dynamics). Locally-linear models are estimated in the reference or healthy system’s reconstructed phase space, and used to quantify the slow drift in the fast systems flow. This quantification is accomplished by comparing evolutions from the same initial point $x_0$, over short-time interval $t_p$, of current fast-time system’s trajectory and a trajectory predicted by the reference model.

In this paper, a mathematical model of modified experimental system used for a scalar damage variable tracking\textsuperscript{39} and prediction\textsuperscript{38} is studied to develop a method for a vector-valued damage variable. In the following section, a brief description of damage tracking and prediction algorithms are given, and their main properties are described. In the next section, a mathematical model for the experimental electro-mechanical system is derived. Application of the tracking algorithm to the simulated beam displacement measurements and failure predictions, based on the estimated battery states, are given in Sec. 4. At the end, we conclude with summary and discussion.
2. DIAGNOSTIC AND PROGNOSTIC PROCEDURES

The essential idea of tracking scalar hidden variables in system’s state space is described in great detail in Ref. 39. Here, a brief description of damage tracking and failure prediction algorithms is presented and extended to handle vector-valued damage variables.

2.1. Slow Variable Tracking Function

Let us consider a general system described by Eq. (1). For \( 0 < \epsilon \ll 1 \) the complete state of Eq. (1) is the direct product \((x, \phi)\). We are interested in tracking changes in the slow variable \( \phi \) characterized by its average value over all \( t \in t_D \), where \( t_D \) is the intermediate time interval corresponding to one data record and \( t_D \ll t_F \). For any \( t_0 \in t_D \) we can write \( \phi = \phi_0 + O(\epsilon) \). Then the state of the fast subsystem can be written as follows:

\[
\begin{align*}
  x(t_0 + t_p; \epsilon) &= X(x_0, \mu(\phi), t_0, t_p), \\
  &= X(x_0, t_0, t_p; \mu(\phi_0)) + O(\epsilon), \\
  &= X(x_0, t_0, t_p; \mu(\phi_R)) + \frac{\partial X}{\partial \mu} \frac{\partial \mu}{\partial \phi} (\phi_0 - \phi_R) + O(||\phi_0 - \phi_R||^2) + O(\epsilon),
\end{align*}
\]

where on the third line we have Taylor expanded the expression about \( \phi_R = \phi(t_R) \), the reference value of the slow variable, for some reference time \( t_0 = t_R \). Then, for fixed \( x_0 \), \( \phi_R \), and \( t_p \), we define a tracking function as follows:

\[
\begin{align*}
  e &= X(x_0, \mu(\phi_0), t_0, t_p) - X(x_0, \mu(\phi_R), t_0, t_p), \\
  &= \frac{\partial X}{\partial \mu} \frac{\partial \mu}{\partial \phi} (\phi_0 - \phi_R) + O(||\phi_0 - \phi_R||^2) + O(\epsilon),
\end{align*}
\]

where the coefficient matrices in the Taylor expansion are evaluated at \( x = x_0 \) and \( \phi = \phi_R \).

In principle, for a fixed \( x_0 \) and \( t_p \), \( e \) can be evaluated for many different values of \( \phi \), and given that the matrix \( \frac{\partial X}{\partial \mu} \frac{\partial \mu}{\partial \phi} \) has maximal rank, one can construct mapping \( e^{-1}(\phi) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) required for true tracking. Therefore, if the function \( e(\phi) \) can be determined experimentally, subsequently it can be used to determine \( \phi \) provided that the observability conditions are satisfied, i.e., if we have \( n > p > m \), in addition to \( \text{Range}(\frac{\partial X}{\partial \mu}) \subset \mathbb{R}^n \) and \( \text{Range}(\frac{\partial \mu}{\partial \phi}) \subset \mathbb{R}^p \) having dimensions \( p \) and \( m \), respectively. If these conditions are met, we expect the output of the tracking function to be an affine transformation of the drifting variable (linear observability):

\[
e = C(x_0, t_0, t_p; \phi_0 + c(x_0, t_0, t_p, \epsilon)),
\]

where \( C = \frac{\partial X}{\partial \mu} \frac{\partial \mu}{\partial \phi} \) is an \( n \times m \) matrix, and \( c = -\frac{\partial X}{\partial \mu} \frac{\partial \mu}{\partial \phi} \phi_R + O(||\phi_0 - \phi_R||^2) + O(\epsilon) \) is an \( n \times 1 \) vector; all terms are evaluated using \( x = x_0 \) and \( \phi = \phi_R \). Please note that even when linear observability fails, higher order observability may in many cases be possible.

In an experimental context, however, the procedure illustrated above is prone to errors. It is usually impossible to repeatedly start the system from the same initial condition. Furthermore, the observability conditions clearly depend on \( x_0, t_0 \), and \( t_p \), and so we might attempt to somehow use many values of \( x_0 \) and/or \( t_0 \) and/or \( t_p \) to deal with this repeatability problem as well as to increase the robustness of the method. The model considered in this paper exhibits a chaotic behavior for a parameter’s rage of interest. It is a well know characteristic of a chaotic trajectory to fill out a large section of phase space. Therefore, the averaging over many values of \( x_0 \) (and \( t_0 \) at the same time, due to ergodicity) is easily possible by following the fast system’s trajectory in the phase space. This is essentially the approach followed here.

To actually calculate the tracking function \( e \) for any given initial condition \( x_0 \), we need to know how the fast subsystem evolves over the time interval \( t_p \) for the current value of \( \phi_0 \), as well as how this subsystem would have evolved for the reference value of \( \phi_R \). Since the system’s fast time behavior for the current value of \( \phi_0 \) is directly measurable (i.e., we can reconstruct the fast dynamics using a sensor measurement from the fast subsystem), the strategy used in this paper is to compare it to the predictions of a reference model describing the fast subsystem behavior for \( \phi = \phi_R + O(\epsilon) \).
The fast-time phase space of Eq. (1a) is reconstructed using delay coordinate embedding. In this procedure the measured scalar time series \( \{x(k)\}_{k=1}^{M} \) are converted to a series of vectors \( y(k) = (x(k), x(k + \tau), \ldots, x(k + (d - 1)\tau))^T \in \mathbb{R}^d \), evolving under some nonlinear map of the form
\[
y(k + 1) = P(y(k); \phi). \tag{5}
\]
The dimension of embedding space \( d \) is determined using the method of false nearest neighbors and delay parameter \( \tau \) is taken to be the first minimum of average mutual information.

The local linear models
\[
y'(k + 1) = A_k y(k) + a_k, \tag{6}
\]
where \( A_k \) is an \( n \times n \) matrix and \( a_k \) is an \( n \times 1 \) vector, are used to approximate Eq. (5) in the reference system’s reconstructed phase space. The parameters of local linear models are determined by calculating the best linear fit between \( N \) nearest neighbors of \( y(k) \) and their future states \( y(k + 1) \) in the reconstructed phase space.

Then the tracking function (Eq. 3) for initial point \( y(k) \) can then be written as
\[
e_1(k) = P(y(k), \phi_0) - P(y(k), \phi_R) = y(k + 1) - A_k y(k) - a_k + E^M(y(k)) = E(y(k)) + E^M(y(k)), \tag{7}
\]
where subscript in \( e_1(k) \) denotes that \( t_p = t_s \), \( E^M(y(k)) \) represents the local linear model error and
\[
E(y(k)) = y(k + 1) - A_k y(k) - a_k \tag{8}
\]
is the estimated tracking function that can be determined experimentally. The use of \( E(y(k)) \) in place of tracking function is justified if \( O(E^M) \) is small compared to \( O(E) \).

### 2.2. Multidimensional Damage Variable Tracking Metric

In previous studies the norm of averaged value of the tracking function over one data record \( \|E(y(k))\| \) provided a good tracking metric for scalar damage variables. In case of multidimensional damage variable, this simple tracking metric is inadequate. The approach taken in this study is to evaluate the average value of estimated tracking function in \( N_e \) disjoint regions \( B_i \ (i = 1, \ldots, N_e) \) of the reconstructed phase space. Consider the following tracking metric for multimode damage variable:
\[
S = \bigcup_{i=1}^{N_e} \|E_i\|^{-1} \sum_{y(k) \in B_i} E(y(k)). \tag{9}
\]
Since \( E \in \mathbb{R}^d \), for each data record we will have total of \( N_s = N_e \times d \) statistics. It is conjectured that there is an affine transformation \( V : \mathbb{R}^{N_s} \to \mathbb{R}^m \) that maps these statistics \( S \in \mathbb{R}^{N_e} \) onto the damage states \( \phi \in \mathbb{R}^m \):
\[
\{\phi\} = VS + v, \tag{10}
\]
where \( \{\phi\} \) denotes the estimated damage state, \( V \) is an \( N_s \times N_x \) matrix and \( v \) is an \( N_x \times 1 \) vector.

If we assume that we have total of \( N_s \) data records, then we can form an \( N_s + 1 \times N_x \) matrix \( W \), such that \( W_i = (S_i^T 1)^T \) for each data record \( i = 1, \ldots, N_r \). Now, if we form an \( m \times N_r \) matrix \( \Psi \), such that \( \Psi_i = (\phi_i) \) is composed the average values of \( \phi \) for each data record \( i = 1, \ldots, N_r \), we can calculate the needed affine transformation in the mean squares sense using:
\[
[V \ v] = \Psi W (WW^T)^{-1}. \tag{11}
\]
The coordinate transformation from \( S \) to \( \{\phi\} \) defined by Eq. (10) can be determined if the independent measurement of actual damage state variables are available. The existence of this transformation can be empirically demonstrated using the mathematical model considered here and in some test experiments where this independent measurements are possible. However, in real life applications one has to use methods similar to optimal tracking described in Ref. 44. Another possibility is to use proper orthogonal decomposition on the \( S \) statistics to identify dominant nonlinear modes and utilize them for tracking. Using simulated model equations we demonstrate that this is indeed possible for the ideal case; however, in experimental procedure the former approach is more robust.
2.3. Multimode Failure Prediction

The tracking procedure described in previous section provides estimates of current state of damage variables. These estimates can be used to both estimate parameters of damage evolution model, as well as predict remaining time to failure. Damage model parameter estimation is not considered in this paper. Here, we assume that the form of the damage evolution model and its parameters are known a priori. Given this, one can derive an expression of the remaining time to failure \( t_F = t(\phi_F) - t \), provided with predefined failure surface \( \phi_F = \phi(t_F) \). Then, for the time to failure estimation procedure the following discrete-time state transition and measurement equations can be used

\[
\begin{align*}
    t_F(l+1) &= t_F(l) - \Delta t + w_F(l+1), \\
    z_F(l+1) &= t_F(l+1) + v_F(l+1),
\end{align*}
\]

where \( \Delta t = M t_s \) represents the time length of consecutive scalar data records sampled at \( t_s \) with no gap between them; \( w_F(l) \) and \( v_F(l) \) are assumed to be white, independent random variable with Gaussian distributions \( p(w_F(l)) \sim N(0,Q_F(l)) \) and \( p(v_F(l)) \sim N(0,R_F(l)) \), respectively. Here, \( R_F(l) = |(dt/\phi)R_\phi(l)| \) is the covariance associated with each measurement, where \( R_\phi(l) \) is the covariance for each estimate of \( \phi(l) \). Since the Eqs. (12) are linear, we use the linear Kalman filter with following time update equations:

\[
\begin{align*}
    \hat{t}_F(l+1) &= \hat{t}_F(l) - \Delta t, \\
    P_F(l+1) &= P_F(l) + Q_F(l).
\end{align*}
\]

The corresponding measurement update equations are:

\[
\begin{align*}
    K_F(l) &= \frac{P_F(l)}{(P_F(l) + R_F(l))}, \\
    \hat{t}_F(l) &= \hat{t}_F(l) + K_F(l)(z_F(l) - \hat{t}_F(l)), \\
    P_F(l) &= (1 - K_F(l))P_F(l).
\end{align*}
\]

3. MATHEMATICAL MODEL DERIVATION

Here, a model of an experimental system in our laboratory is considered. The original system was a constrained version\(^{45}\) of a vibrating beam in the force field of two permanent magnets.\(^{46}\) The difference with the previous system is that both permanent magnets are augmented by the electromagnets (see Fig. 1). The force field at the beam tip drifts as the batteries powering electromagnets discharge. For the model parameters, it is assumed that a complete discharge of the batteries manifest itself in about 3.5% change in natural frequencies of small oscillations in each electromagnet well.

The experimental system can be viewed as a mechanical subsystem, Fig. 3(a), coupled with an electromagnetic subsystem, Fig. 3(b). The coupling has the following effects: as the battery discharges, the decrease of “stiffness” in the potential wells of the electromagnets lowers the natural frequency of small oscillations in those wells by 3.5%; meanwhile, the beam motions parametrically excite the electromagnetic subsystems. For this study, both electromagnet circuits are assumed identical.

A lumped parameter model for the experimental system with a single mechanical degree of freedom can be developed using Lagrange’s equations. For simplicity, the effects of gravity is neglected; this introduces only a small nonlinear error term because the first-order effect of the gravity can be incorporated into the torsion spring constant \( k \), shown in Fig. 3(a). In addition, energy losses due to eddy currents in the beam are ignored, assuming that all losses can be modeled using just the torsional damping coefficient \( c \) and circuit resistances.

![Figure 1: Schematic of the experimental system](image-url)
This derivation follows one presented previously\textsuperscript{47}; the main difference is the consideration of two electromagnets instead of one. Therefore, we will not dwell on the details and present the final form of dimensionless equations of motion, after brief description of main parts of the model.

![Diagram](image)

Figure 2. Lumped parameter model of the experimental system: (a) single degree of freedom model of the constrained beam; (b) schematic of the electromagnetic subsystem; (c) magnetic restoring force at the beam tip

The model for the electromagnetic subsystems is shown schematically in Fig. 3(b), where: $\Phi(\epsilon t)$ ($\epsilon \ll 1$) is the slowly drifting battery (open circuit) voltage, taken to be its internal state; $R_i$ is the internal resistance of the battery; $R_e$ is the external resistance of the circuit; and $L$ is the inductance of the electromagnet. The inductance $L$ is a function of the position described by following inductance model for $i = 1, 2$ electromagnets:

$$L_i(\theta) = L_0 \left(1 + \frac{L_r}{1 + \kappa (\theta - \lambda_i)^2}\right),$$

where $L_0$, $L_r$ and $\kappa$ are positive constants, and $\lambda_i$ is defined later for each electromagnet.

The shape of the potential energy field due to the permanent magnets at the beam’s end is described using a double-well potential. The simplest form of force field due to this potential, which is adopted here, is a symmetric cubic polynomial

$$-Q(\theta) = a_3 \theta^3 - a_1 \theta,$$

where, $a_1$ and $a_3$ are positive constants and in Eq. (15) we use $\lambda_1 = \sqrt{a_1/a_3}$ for the electromagnet in front and $\lambda_2 = -\sqrt{a_1/a_3}$ for the electromagnet in back.

Now letting $-ml\ddot{z} = F \cos \omega t$ and $\omega_n^2 = k/ml^2$, rescaling time and defining dimensionless battery voltage and current variables by

$$t \rightarrow \omega_n t, \quad \phi_i = \sqrt{\frac{ml^2 \Phi_i}{L_0}} \frac{\Phi_i}{k}, \quad \text{and} \quad \psi_i = \sqrt{\frac{L_0}{ml^2}} I_i \quad (i = 1, 2),$$

dimensionless equation of motion for the beam vibration,

$$\ddot{\theta} + \mu \dot{\theta} + (1 - \alpha_1) \theta + a_3 \theta^3 + \sum_{i=1}^{2} \frac{\kappa L_r (\theta - \lambda_i)}{1 + \kappa (\theta - \lambda_i)^2} \psi_i^2 = f \cos \Omega t,$$

is obtained. Eq. (18) is coupled to a set of equations (for $i = 1, 2$) describing the current flow in the electromagnets’ circuits:

$$\left[1 + \frac{L_r}{1 + \kappa (\theta - \lambda_i)^2}\right] \dot{\psi}_i + \left[r - \frac{2 \kappa L_r (\theta - \lambda_i) \dot{\theta}}{1 + \kappa (\theta - \lambda_i)^2}\right] \psi_i = \phi_i,$$

where

$$\mu = \frac{c}{\omega_n}, \quad \alpha_j = \frac{a_i}{\omega_n^2} \quad (j = 1, 3), \quad f = \frac{F}{\omega_n^2}, \quad \Omega = \frac{\omega}{\omega_n}, \quad r = \frac{R_i + R_e}{\omega_n}, \quad \text{and} \quad \phi = \frac{\Phi}{\omega_n^2}.\quad (20)$$

The time evolution of the battery voltage is governed by electrochemical processes which is not explicitly modeled. Instead, given the experimental battery voltage evolution trends typically seen in the experiments\textsuperscript{39} we simply use the following voltage evolution law for both batteries,

$$\dot{\psi}_i = -\epsilon_i (\phi_i - \xi) \left(1 + \gamma (\phi_i - \eta)^2\right) \quad (i = 1, 2),$$

(21)
where $\xi$, $\gamma$ and $\eta$ are positive constants, and the rate constant $\epsilon_i$ satisfies $0 < \epsilon_i \ll 1$.

From the experimental results, as well as other published results, we know that the internal battery voltage initially drops rapidly to the operating range and remains there for an extended period, slowly decreasing before another rapid drop to near zero voltage at the end of the battery’s life. The form of Eq. (21) is the simplest rate law which will exhibit these characteristics.

4. SIMULATION OF THE MODEL

In this section numerical simulations of the complete system of Eqs. (18), (19) and (21), which were carried out to investigate the performance of the tracking technique are described. The equations were integrated numerically with a standard 4th-order variable-step-size Runge-Kutta algorithm. Since this study was inspired by the experimental investigation of scalar damage tracking method, the parameters for the model were selected to match the properties of the experimental system in key ways, as described below.

It is assumed that fully charged batteries provide 9 V DC power. It is also assumed that the natural frequency of small oscillations in the potential well with the electromagnet is changes by 3.5% going from 8.8 Hz to 8.5 Hz. The effective mass $m=0.2$ kg, length $l=0.128$ m, were obtained directly from the experiment. A fourth order polynomial fit to a histogram of the experimental reference data was used to estimate $\alpha_1=2.6558$ and $\alpha_3=0.8805$. The effective damping parameter was assumed to be $\mu=0.088$.

By linearizing Eqs. (18) and (19) about the stable equilibria $(\theta = \pm \sqrt{(\alpha_1 - 1)/\alpha_3})$ for $\Phi = 0$, expressions for the frequency of small oscillations were obtained, which were used to estimate the effective stiffness $k$ using Eq. (20). Since the frequency for $\Phi = 0$ in both electromagnet’s well is 8.5 Hz, one can estimate $k = 0.071$ using $m$, $l$, and $\alpha_1$. We choose $\kappa=0.746$ so that the effect of $L_r$ on inductance amplitude decreased to 10% at 2A distance. To determine other parameters the case of $\Phi_1 = 9$ and $\Phi_2 = 0$ was considered, for which the natural frequency in the powered electromagnet’s well is 8.8 Hz. Using this information one finds $L_0 = 0.079$ after arbitrarily setting $r=10$. Forcing amplitude $f = 1$ and forcing frequency $\Omega = 1.95$ were chosen so that the system exhibited nominally chaotic motion throughout the experiment\(^{†}\).

Other parameters used in the simulations were: $\eta = 22$, $\gamma = 1$ and $\xi = 0.1$. The rate parameters for battery evolution laws were chosen to be $\epsilon_1 = 1 \times 10^{-6}$ and $\epsilon_2 = 0.5 \times 10^{-6}$, so that the first battery discharged twice as fast as the second. Using the above parameters, one has $\Phi = 2.846 \Phi$, and the observed fast-time dynamics of the simulated $\theta$ were found to have qualitatively similar trajectories in the $(\theta, \dot{\theta})$ phase space to those observed with experimental strain-gauge time series.

4.1. Tracking Function Calculation

For this calculation we have randomly selected 64 initial points in the fast-time system’s, Eq. (18), phase space. The evolution of $\phi_1$ and $\phi_2$ were calculated using above parameters and Eq. (21). The the tracking function was evaluated for 300 different values of $\phi_1$ and $\phi_2$ uniformly spaced in time and $t_p = T_s = 0.1$. For further calculations we have only used the first component of the evaluated vector-valued tracking function. Fig. 3 shows the first two proper orthogonal modes of the tracking function. Subplot in this figure shows the first six largest singular values of a 64 $\times$ 300 matrix constructed out of the tracking function evaluate for 64 initial conditions.

For the analytically calculated tracking function statistics we have a clearly dominant first two nonlinear modes. For a matrix $S$ we used only this two modes and determined the needed affine transformation using Eq. (11). Fig. 4 shows the results of this calculation that are plotted on top of the actual evolutions of battery voltage. The fit is accurate to the order of 0.01% of battery voltage amplitude as shown in the subplot. Exactly same results would be obtained if we have used any two independent statistics of the tracking function. This results demonstrate that our basic tracking idea is working, however, we still need to show that this procedure can be used in the experimental context as described in the following section.

\(^{†}\)No tests for chaos were performed for these numerical experiments, since the existence or nonexistence of chaos is not particularly relevant to the task at hand. In any case, given the nonstationarity of the fast subsystem, the system can at most be treated as only nominally chaotic.
Note that in Fig. 4 of the tracking curves are plotted on a quadratic scale of $\phi$. For our model, the tracking is linearly related to the square of the battery voltage variables. This is a consequence of the physics included in our model: in Eq. (18), $\phi$ enters into the fast subsystem via an inductive coupling term, and hence only as $\phi^2$. Hence the Taylor expansion Eq. (2) in should be in terms of $\phi^2$, not just $\phi$.

4.2. Damage State Estimation

The simulation used the complete system of Eqs. (18), (19) and (21): $3 \times 10^6$ points were collected with a sampling time of $t_s = 0.1$. At the end of the simulation, both $\phi_1$ and $\phi_2$ reached a $\xi$ value. After waiting an initial period to allow transients to die off, the first $2^{15}$ data points of the scalar $\theta$ data set were used for the reference model. Average mutual information and false nearest neighbors algorithms were used to select a delay $\tau = 7$ sample steps, and an embedding dimension $d = 6$ for the reference data set. The average pointwise dimension of the reference data set was $d_p = 2.7$, supporting our assumption of nominally chaotic system.

For the tracking metric $S$, Eq. (10), the entire reconstructed data set was divided into consecutive data records of size $2^{14}$ points each (i.e., $M = 2^{14}$), and 16 nearest neighbors were used for the reference model construction. The reconstructed reference phase space was partitioned into 16 disjoint regions uniformly distributed along one of the coordinates. The tracking metric was determined by evaluating the mean of the tracking function in each of these partitions. Fig. 5 plots components of $S$ for the coordinate used in partitioning versus time in $10^6$ samples. The singular value decomposition of $S$ statistics had six dominant eigenvalues. Corresponding nonlinear modes plotted versus time in $10^6$ samples are shown in Fig. 6. There is discernable similarity between the first two experimental modes and the modes determined analytically (Fig. 3). However, we have some large local fluctuations in the metric that are the results of change in the initial point population from data record to record. This change is the result of our system going through bifurcations as hidden parameters drift. In particular, a large section of periodic window was observed around $2.7 \times 10^6$ time samples as evident in Figs. 5–6.

The affine transformation Eq. (10) was determined using Eq. (11). Results of this calculation are plotted in Fig. 7 over the actual battery voltage data. The error in the experimental estimates went up to $2.0\%$ of the battery voltage amplitude, which is still low but several orders higher than the error for analytically calculated estimates. The accuracy of these estimates can be further improved by accounting for the accuracy of local linear model when calculating $S$, Eq. (9), as shown in Ref. 39. However, we leave this for the future development. The tracking result shows that using simulated experiment the tracking algorithm accurately recovers (to within an affine transformation) the theoretical tracking curves as shown in Fig. 7. This confirms our conjecture that statistics $S$ can be mapped onto the damage state variables $\phi$, Eq. (10).
The damage tracking method based on measuring, in system’s reconstructed phase space, changes in fast-time system’s flow over short-time intervals due to the slow drift in damage state variables was presented. A mathematical model was derived for an electromechanical system where a forced cantilever beam oscillates in a potential filed formed by two permanent magnets augmented with electromagnets powered by batteries. The force field at the beam tip is perturbed by the battery discharge process. Thus, hidden or damage variables
for this system were two battery states (open circuit battery voltages). The basic tracking idea was validated by analytically calculating battery voltage tracking function for the model considered. Using the mathematical model, it was explicitly demonstrated, through numerical experiment, that the phase space based damage tracking method provides accurate estimates of multidimensional damage variables. This tracking was accomplished using only the simulated data for the beam displacement variable. The estimated battery voltage data was used to obtain a measurement data for the recursive linear filter for the remaining time to failure estimation. It was also shown that the proposed method gives accurate estimates of remaining useful life for both battery discharge processes throughout the whole experiment. The predictions were accurate $1.3 \times 10^6$ time samples ahead of failure for the first battery and $2.6 \times 10^6$ time samples ahead for the second.

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