ABSTRACT
The hidden variable tracking method developed by Cusumano, et al. (1998) is analyzed using a simple model of the mechanical oscillator with nonstationary two-well potential previously studied experimentally. Numerical experiments conducted using the model are in good agreement with the experimental study, and explicitly show how the tracking metric (or drift observer) is related to drift in system parameters caused by the slow evolution of a hidden variable. Using the idea of averaging, the slow flow equation governing hidden variable evolution is obtained. It is shown that solutions to the slow flow equation correspond to the drift trajectory obtained with the experimental tracking method.

INTRODUCTION
Tracking hidden processes in slowly evolving dynamical systems is an important current problem in applied engineering research. This class of problems has direct applications in condition based maintenance and failure prediction of engineering systems. For example, a crack in a spinning shaft may slowly grow over hundreds of thousands of revolutions: online health monitoring and failure prediction for such a system requires estimation of the current state of the damage in the system, often using only shaft vibration data which occurs over much shorter time scales.

The problem is further complicated by the fact that damage evolution processes are often hidden from an observer, i.e., are not directly accessible. Sometimes, the proper damage state variable is not even known. In such cases there is a need for methods of tracking the hidden damage evolution processes using only the directly accessible and physically measurable quantities. In this context, one can think of the problem of tracking damage as equivalent to tracking parameter drift in a directly observable subsystem.

Cusumano et al. (1998) developed an experimental method for tracking slowly evolving hidden processes in dynamical systems. The method implicitly treats the damage, or parameter drift, as evolving in a hierarchical dynamical system consisting of a “fast-time,” directly observable, subsystem coupled to a “slow-time” subsystem with the form

\begin{align}
\dot{x} &= f(x, \mu(\phi), t), \\
\dot{\phi} &= \epsilon g(x, \phi, t),
\end{align}

where \(x \in \mathbb{R}^l\) is the directly observable fast dynamic variable subsystem Eq. (1a); \(\phi \in \mathbb{R}^m\) is the slow dynamic variable (“hidden” damage state) of subsystem Eq. (1b); the parameter vector \(\mu \in \mathbb{R}^n\) is a function of \(\phi\); \(t\) is time; and the rate constant \(0 < \epsilon \ll 1\) defines the time scale separation between the fast dynamics and the slow drift. Note that \(\epsilon = 0\) corresponds to the constant parameter case \((\mu = \mu(\phi_0) \equiv \mu_0)\).

In the experimental procedure, the phase space of the fast subsystem is reconstructed using fast-variable data recorded over intermediate time scales (times which are long compared to the fast dynamics but short compared to drift dynamics). Locally-linear models are estimated in the reconstructed phase space, and used to quantify the slow drift in the fast dynamics.

In Cusumano et al. (1998), the tracking procedure was successfully applied to an experimental system consisting of a forced mechanical oscillator with a nonstationary two-well potential. As shown in Fig. 1, the system consisted of a...
beam, constrained using stiffeners to essentially one degree of freedom, and with two magnets near its free end. The beam was mounted on a shaker and forced sinusoidally. A small perturbation in one magnetic potential energy well was introduced by a battery powered electromagnet placed on top of the permanent magnet. In the experiment, the battery state was taken to be “hidden” and the data analysis was done using only the scalar strain time series from the beam. The voltage $V$ across the battery terminal was simultaneously recorded. It was shown that the parameter drift tracking algorithm could follow the battery discharge as represented by a local (i.e., moving) average of $V$.

In the experimental system, the observable terminal voltage $V$ was inductively coupled to motions of the beam and hence could not be treated as a direct measurement of the internal battery state. In the earlier study, no direct analytical relation was established between the tracking algorithm and the “damage state variable” (the open circuit battery voltage). In this paper, a mathematical model for the experimental system is developed. Using this model, we apply numerical integration, analysis, and the tracking algorithm to theoretically and explicitly demonstrate the connection between the slow drift process and the tracking algorithm applied to the fast time data.

An unconstrained beam in the force field of two magnets was previously studied by Moon and Holmes (1979, 1980). They demonstrated that Duffing’s equation provides a simple model for the steady state motions of the fast system (i.e. without drift in the magnetic forces). Kimble and Cusumano (1996) used a similar model to validate experimental studies of the stochastic interrogation method; in that work, the beam was constrained as it is here. Thus, previous analytical and experimental studies have used the system employed as the basis for this work, but only for the non-drifting case.

In the next section we derive a mathematical model for the experimental electro-mechanical system, incorporating the slow drift in system parameters. Then, the tracking method as used in the experimental studies is summarized and applied to the output from a simulation of the analytical model. Finally, we demonstrate the relationship of the tracking algorithm with the slow flow equation, obtained through averaging, that approximately describes the evolution of the battery state.

**ANALYTICAL MODEL**

The experimental system can be viewed as as a mechanical subsystem (Fig. 2) coupled with an electromagnetic subsystem (Fig. 3). The coupling has the following effects: as the battery discharges, the decrease of “stiffness” in the potential well of the electromagnet lowers the natural frequency of small oscillations in that well by $\approx 5\%$; meanwhile, the beam motions parametrically excite the electromagnetic subsystem.

A lumped parameter model for the experimental system, with a single mechanical degree of freedom, can be developed using Lagrange’s equations. For simplicity, we neglect the effects of gravity; this introduces only a small nonlinear error term because the first-order effect of the gravity can be incorporated into the torsion spring constant $k$ in Fig. 2. In addition, we ignore energy losses due to eddy currents in the beam, assuming that all losses can be modeled using just the torsional damping coefficient $c$ and circuit resistances.

The shape of the potential energy field due to the permanent magnets at the beam’s end is described using a double-well potential. The simplest form for such a potential ($P$), which we adopt here, is a fourth-order polynomial,

$$P(\theta) = b_4 \theta^4 + b_3 \theta^3 + b_2 \theta^2 + b_1 \theta + b_0. \quad (2)$$

The model for the electromagnetic subsystem is shown schematically in Fig. 3, where: $\Phi(\epsilon t) \ (\epsilon \ll 1)$ is the slowly
drifting battery (open circuit) voltage, taken to be its internal state; \( R_i \) is the internal resistance of the battery; \( R_e \) is the external resistance of the circuit; and \( L \) is the inductance of the electromagnet.

The inductance \( L \) is a function of the position of the beam. A simple model which is adequate for our purposes is given by

\[
L(\theta) = \frac{L_0}{1 + \kappa (\theta - \lambda)^2},
\]

where, \( L_0, \kappa, \) and \( \lambda \) are positive constants.

Using a charge formulation (Woodson and Melcher, 1968) for the electromagnetic subsystem, the Lagrangian for the complete system can be expressed as

\[
\mathcal{L} = \frac{1}{2}m (\dot{z} + l \dot{\theta})^2 + \frac{1}{2}L(\theta) \dot{q}^2 - \frac{1}{2}k\theta^2 - P(\theta).
\]

Here, \( m \) is the (effective) end mass, \( z(t) \) is the imposed base excitation, \( q \) is the net charge that has flowed out of the battery up to the instant of interest. The overdot denotes differentiation with respect to time.

The virtual work associated with nonconservative generalized forces (damping and battery voltage) can be written as

\[
\delta W = -c \dot{\theta} \delta \theta + (\Phi - R \dot{q}) \delta q,
\]

where \( R = R_i + R_e \) is the total resistance of the circuit. Using Eqs. (4) and (5), Lagrange’s equations yield:

\[
ml^2 \ddot{\theta} + c \dot{\theta} + k \theta + \frac{\partial P}{\partial \theta} - \frac{1}{2} \frac{\partial L}{\partial \theta} \dot{q}^2 = -ml \ddot{z}, \quad (6a)
\]

\[
L \ddot{q} + \left( \frac{\partial L}{\partial \theta} \dot{\theta} + R \right) \dot{q} = \Phi. \quad (6b)
\]

We assume for simplicity that \( P \) is symmetric, \( i.e. b_3 = b_1 = 0 \) in Eq. (2). The force \( Q \) at the beam tip is then given by the cubic polynomial

\[
\frac{\partial P}{\partial \theta} = -Q = a_3 \theta^3 - a_1 \theta,
\]

where, \( a_3 = 4b_4 \) and \( a_1 = -2b_2 \) are positive constants. In Eq. (3) we use \( \lambda = \sqrt{a_1/a_3} \) so that \( L = L_0 \) when the tip of the beam is at the static equilibrium directly above the electromagnet.

We set \(-ml\ddot{z} = F \cos \omega t\) and define the new parameters

\[
\alpha_i = \frac{a_i}{k} \quad (i = 0, \ldots, 3),
\]

\[
f = \frac{F}{k} \quad \omega = \omega_n, \quad \text{and} \quad r = \frac{R}{L_0 \omega_n}.
\]

Then, rescaling time and defining dimensionless battery voltage and current variables by

\[
i = \omega_n t, \quad \phi = \sqrt{\frac{m l^2}{L_0}} \frac{\Phi}{k} \quad \text{and} \quad \psi = \sqrt{\frac{L_0}{m l^2}} \frac{dq}{dt}, \quad (9)
\]

Eqs. (6) can be written (after dropping the overbar from \( \ddot{q} \) ) in dimensionless form as:

\[
\dot{\theta} + \mu \dot{\theta} + (1 - a_1) \theta + \alpha_3 \theta^3 + \frac{\kappa (\theta - \lambda)}{(1 + \kappa (\theta - \lambda)^2)^2} \psi^2
\]

\[
= f \cos \Omega t, \quad (10a)
\]

\[
\frac{1}{1 + \kappa (\theta - \lambda)^2} \psi + \left[ \frac{2 \kappa (\theta - \lambda)}{(1 + \kappa (\theta - \lambda)^2)^2} + r \right] \psi = \phi, \quad (10b)
\]

where the overdot now indicates differentiation with respect to the rescaled time as defined by Eqs. (8) and (9).

The time evolution of the battery voltage is governed by electrochemical processes which we do not explicitly model. Instead, given the experimental battery voltage evolution trends typically seen in the experiments (see Cusumano et al., 1998) we simply use the following voltage evolution law

\[
\dot{\phi} = -\epsilon \psi \left( 1 + \gamma (\phi - \eta)^2 \right), \quad (11)
\]

where \( \gamma \) and \( \eta \) are positive constants, the rate constant \( \epsilon \) satisfies \( 0 < \epsilon \ll 1 \).

The battery model of Eq. (11) is the simplest rate law capable of matching the qualitative characteristics of the battery discharge curves observed experimentally. It incorporates a linear dependance of the battery voltage drift rate on the current \( \psi \) in the circuit, which we expect to be reasonably accurate under the conditions seen in the experiments, since the external resistor \( R_e \) was chosen so that the current out of the battery was always positive. Previous experimental results, as well as the results of other researchers (see, for example, battery discharge curves in Strauss, et al. 1999), show that the battery voltage drops
rapidly to the operating range and remains there for an extended period, slowly decreasing before another rapid drop to near zero voltage at the end of the battery’s life.

Equations (10) and (11) give a complete description of our system. Note that, in terms of the general hierarchical system description of Eqs. (1), $\theta$, $\dot{\theta}$ and $\psi$ are the fast state variables (i.e., $x = (\theta, \dot{\theta}, \psi) \in \mathbb{R}^3$), and the open circuit battery voltage $\phi$ is the hidden, slowly drifting “damage” variable ($\phi = \phi \in \mathbb{R}$).

For the analysis presented later, it is convenient to simplify our system further by making the following additional approximation. If we assume that the variable inductance coefficient $\kappa$ in Eq. (3) is small ($\kappa \ll 1$), and that $r = O(1)$ with respect to $\kappa$, then the current equation Eq. (10b) becomes

$$\dot{\psi} = \phi - r\psi + O(\kappa).$$

The dimensionless parameter $r$ can be interpreted as the ratio of the relaxation time scale for the electromagnetic subsystem ($L_0/R$) to the natural oscillation period of the mechanical subsystem, ($\propto 1/\omega_n$). In the experimental system, these two time scales were comparable, so it is reasonable to approximate $\psi$ by its “quasi-steady state” value

$$\psi \approx \frac{\phi}{r},$$

in the mechanical subsystem Eq. (10a) and battery evolution equation (11). This gives the reduced system

$$\ddot{\theta} + \mu \dot{\theta} + (1 - \alpha_1) \theta + \alpha_3 \dot{\theta} + \frac{\kappa (\theta - \lambda)}{(1 + \kappa (\theta - \lambda))^2} \left( \frac{\phi}{r} \right)^2 = f \cos \Omega t,$$  \hspace{1cm} (14a)

$$\dot{\phi} = -\epsilon \frac{\phi}{r} \left( 1 + \gamma (\phi - \eta)^2 \right),$$ \hspace{1cm} (14b)

with (referring once again to Eqs. 1) fast-time state variable $x = (\theta, \dot{\theta}) \in \mathbb{R}^2$, and slow-time variable $\phi = \phi \in \mathbb{R}$.

Before discussing the simulation results for this model, a brief summary of the tracking algorithm is now presented, both in the interest of completeness, and to motivate the subsequent analysis.

**TRACKING PROCEDURE**

To motivate the experimental tracking procedure, consider the case where $\epsilon = 0$ in Eq. (1b), so that we consider the system of Eq. (1a) with fixed $\phi$. Imagine the system has a solution with $x = x_0$ at $t = t_0$, so that at time $t_0 + T$ the state of the system is $x(t_0 + T) = X(x_0, t_0 + T; \mu(\phi))$. Fixing $t_0$, $T$ and $x_0$, $X$ is purely a function of $\mu(\phi)$, say $X(x_0, t_0 + T; \mu(\phi))$. In the experimental context, however, the procedure outlined is impractical because it usually impossible to repeatedly start the system from the same initial condition.

**Fig. 4:** Schematic drawing illustrating the basic ideas about tracking in state space. Trajectories are shown as black lines. The dashed line represents an observer function $h(\mu(\phi))$.

$h(\mu(\phi))$, that describes an $n$-dimensional surface in $\mathbb{R}^l$ (we assume that $n \leq l$) as $\mu$ varies.

Now, imagine several experiments where the system is repeatedly started with the same $t_0$ and $x_0$, but with slightly different values of $\phi$ (see Fig. 4). In this way, in principle at least, the tracking function $h(\mu(\phi))$ can be determined experimentally, and subsequently used to detect changes in $\mu(\phi)$.

For sufficiently small net changes in $\mu(\phi)$ about some initial value $\mu(\phi_0)$ we write

$$\Delta h = h(\mu(\phi)) - h(\mu(\phi_0)) = Dh(\mu(\phi_0)) \Delta \mu + O(\|\Delta \mu\|^2)$$ \hspace{1cm} (15)

where $Dh$ denotes the derivative of $h$ with respect to its argument. If the emphasis in the problem is on following changes in $\mu$, we say that our aim is parameter tracking. Under reasonable observability conditions (related to the derivatives of $\mu$ with respect to $\phi$), successful tracking of $\mu$ can be used to estimate the state of $\phi$, in which case the same algorithm can be used for hidden variable tracking.

In the system considered here, we can take

$$\mu(\phi) = \mu(\phi) = \kappa \left( \frac{\phi}{r} \right)^2,$$ \hspace{1cm} (16)

that is, $m = n = 1$ in Eqs. (1). Using Eqs. (14) derived in the previous section, the tracking function $h$ can be computed numerically starting from any particular initial point ($\theta_0, \dot{\theta}_0$) in the phase space of the fast system. The results of one such calculation are shown in Fig. 5 (for the simulation, we used parameters given in the following section). A point was randomly picked in the phase space, and its state at some time $T$ later was calculated for different starting values of $\phi$. The figure shows, as expected, that $\Delta h$ is a smooth function of $\phi$ for reasonable values of $T$. For parameter (or hidden variable) tracking in an experimental context, however, the procedure outlined is impractical because it usually impossible to repeatedly start the system from the same initial condition.
In order to reduce errors and improve robustness, we attempt to use many values of $x_0$. For a general nonlinear and non-autonomous system, it is difficult to examine values of the tracking function at many values of $x_0$. However, if the system is chaotic, then averaging over many values of $x_0$ is easily possible simply by following the fast system’s trajectory as it moves over the chaotic attractor. With this motivation, we average $h$ over data points collected over intermediate time scales (short compared to drift dynamics, long compared to fast dynamics). The rationale is that the average is computed over a large enough number of points to obtain an accurate estimate of $\Delta h$ averaged over the attractor, yet rapidly enough that drift during the averaging process has a small effect.

Now, we describe how the change in the tracking function, $\Delta h$, is actually estimated in the experimental procedure. Let us say that the fast system is at a state $x_0$ at some instant $t_0$; and that we have picked a suitable $T$. Then, in order to calculate $\Delta h$, we need to know both how the system evolves over a time interval $T$ for the current value of $\phi$, as well as how the system would have evolved under the reference value $\phi_0$. From experiment, the evolution under the current $\phi$ is directly measurable. However, the evolution that would have occurred under $\phi_0$ needs to be predicted using a reference model, which we now discuss.

Construction of Reference Model: The reference model should tell us how the reference system evolves starting from any initial condition in the phase space. One way to construct such a model is to repeatedly restart the reference system from many different initial conditions (e.g., by stochastic interrogation, as in Cusumano and Kimble, 1995) that together cover the region of interest in the phase space.

However, we study a nonlinear oscillator in a chaotic regime, at least in the reference state (thus, a large portion of the phase space is occupied by the attractor, and traversed by the system trajectory). We measure a scalar time series corresponding to one of the “fast” state variables of this system (1a), sampled at equal time intervals. For our purposes, it is simple to construct the reference model directly from a single scalar time series, using delay coordinate embedding (see Takens, 1981, and Sauer et al., 1991).

Accordingly, the scalar time series is used to reconstruct a phase space of appropriate dimension (say, $d$) for the system. The scalar time series is converted to a series of vectors $y \in \mathbb{R}^d$, evolving under a map of the form $y(l+1) = P(y(l); \phi)$. We examine the $k^{th}$ iterate of $P$, i.e.,

$$y(l + k) = P^k(y(l); \phi).$$ (17)

The observation period $T$ discussed earlier now corresponds to picking an integer $k$.

That is, given a scalar time series $\{x(n)\}_{n=1}^M$, where $M$ is the sample size, we create vectors $y(n)$ of the form:

$$y^T(n) = (x_1(n), x_2(n), \ldots, x_d(n))$$ (18)

where $x_k(n) = x(n + (k - 1)\tau)$ and $\tau \in \mathbb{N}$ is the time delay (see Fig. 6). The time delay is chosen to be the first minimum of the average mutual information (Fraser and Swinney, 1986), and the method of false nearest neighbors (Kennel et al., 1992) is used to determine the minimum
required $d$.

**Local Linear Models:** Working in the global reconstructed phase space of the dynamical system, local models which show how neighborhoods about each data point are mapped forward in time are estimated. The simplest such model is a local linear model relating the state at time $n$, $y(n)$ to the state $y(n + k)$ at time $n + k$:

$$y(n + k) = A^k_n y(n) + a^k_n = B^k_n \hat{y}(n),$$

where the model parameter matrix $A^k_n$ and parameter vector $a^k_n$ are determined by regression at each point in the data set, and where

$$\hat{y}(n) = [y^T(n), 1]^T,$$

and $B^k_n = [A^k_n, a^k_n] = [b^k_{pq}]$. (20)

Basing our model on $N$ nearest neighbors $(y^r(l), r = 1, \ldots, N)$ of the point of interest $(y(l))$ we can determine the optimum model parameters by minimizing

$$\mu_k^2 = \sum_{r=1}^{N} w^r(l) \left| y^r(l + k) - B^k_l y^r(l) \right|^2$$

$$= \sum_{r=1}^{N} w^r(l) \sum_{n=1}^{d} \left( y^r_{nm}(l + k) - \sum_{m=1}^{d+1} b^K_{nm} \hat{y}^r_m(l) \right)^2,$$ (21)

where $w^r(l)$ is a weighting function of our choice. In this study $w^r(l) = \exp(-d_r^2)$, where $d_r$ is the distance from $y(l)$ to $y^r(l)$. Minimizing $\mu_k^2$ with respect to each $b^{K}_{pq}$ ($l$ and $k$ are fixed) we obtain

$$\sum_{r=1}^{N} w^r(l) y^r_{pq}(l + k) \hat{y}^r_q(l) = \sum_{r=1}^{N} w^r(l) \sum_{m=1}^{d+1} b^K_{pm} \hat{y}^r_m(l) \hat{y}^r_q(l).$$ (22)

It is then easy to show that,

$$B^k_l = \left[ \begin{array}{c} \hat{y}^1(l) \\ \hat{y}^2(l) \\ \vdots \\ \hat{y}^N(l) \end{array} \right] T \left( \begin{array}{c} \hat{y}^1(T) \\ \hat{y}^2(T) \\ \vdots \\ \hat{y}^N(T) \end{array} \right) ^{-1},$$ (23)

where

$$\hat{y}_l = \left[ \begin{array}{c} y^1(l) \\ y^2(l) \\ \vdots \\ y^N(l) \end{array} \right] ,$$ (24a)

$$W \hat{y}_l = \left[ \begin{array}{c} y^1(l) \hat{y}^1(l) \\ y^2(l) \hat{y}^2(l) \\ \vdots \\ y^N(l) \hat{y}^N(l) \end{array} \right] ,$$ (24b)

$$\hat{y}_l = \left[ \begin{array}{c} \hat{y}^1(l) \\ \hat{y}^2(l) \\ \vdots \\ \hat{y}^N(l) \end{array} \right] .$$ (24c)

**Tracking Procedure:** We can now summarize the tracking procedure.

First we use reference data to construct a reference model (local linear). At any later stage, we compare the actual behavior of the fast system to that predicted by the reference model. Thus, the model is only implicitly constructed in this approach. These ideas are shown schematically in Fig. 7.

Note that, in the reference system, the initial point $y(l)$ would have been mapped to a “true” point $y^R(l + k)$. However, since we have an imperfect reference model, it actually maps this point to some point $y^M(l + k)$, which is hopefully close to $y^R(l + k)$.

We now introduce the following “error” vectors (for clarity, the dependence of the $E$’s on $l$ is suppressed in the figure):

$$E^R_R(l):=y(l + k) - y^R(l + k), \text{ true error},$$ (25a)

$$E^R_k(l):=y(l + k) - y^M(l + k), \text{ estimated error},$$ (25b)

$$E^M_k(l):=y^R(l + k) - y^M(l + k), \text{ modeling error.}$$ (25c)

The tracking vector $\Delta h$, introduced earlier, is in this case the true error vector $E^R_k(l)$. Since we cannot determine it directly, we use the estimated error vector $E^R_k(l)$ in its place. In general, this is justified if the modeling error $E^M_k(l)$ is small compared to the true error $E^R_k(l)$.

Using these local models, and using an $M$-point sample from the scalar time series collected experimentally at some later stage (i.e., for a slightly changed system), we can estimate the error vector $E^R_k(l)$ of Eq. (25). Since, in this study, we focus on a system with only one changing parameter, we look for a single scalar measure of the set of all values of the estimated error vector. Specifically, we consider the root...
mean square magnitude of $E_k(l)$ over all data points,

$$e_k = \left( \frac{\sum_{l=1}^{M} q(l) |E_k(l)|^2}{\sum_{l=1}^{M} q(l)} \right)^{\frac{1}{2}}, \quad (26)$$

where $q(l)$ is a weighting function. As discussed earlier, this actually corresponds to an average over many values of $t_0$ which is equivalent to a probability-weighted average over the current “attractor” (neglecting the drift occurring during the collection of the $M$-point sample).

We use

$$q(l) = \frac{1}{r_i^{d_f}}, \quad (27)$$

where $r_i$ is the distance to the farthest of the $N$ nearest neighbors of $y(l)$, and $d_f$ is the estimated dimension of the reference attractor (i.e., box counting dimension). The rationale behind this choice is that points with more accurate local fits should be weighted more than points with less accurate local fits.

**SIMULATION OF THE MODEL**

We now describe numerical simulations of the complete system of Eqs. (10-11), which were carried out to investigate the tracking technique and validate experimental results. The equations were integrated numerically with a standard 4th-order variable-step-size Runge-Kutta algorithm provided in Matlab 5.2. The sampling time was 0.1 units (nondimensional). The parameters of this system were not explicitly identified from experiments; however, the simulation qualitatively matched the experiment in some important ways.

In choosing the parameters, we made sure that the change in the natural frequency of small oscillations in the potential well with the electromagnet was 5% (as was the case with the experimental system); and the ratio of the forcing frequency and the system’s first natural frequency was the same as for the experimental system. The coefficients $\alpha_1$ and $\alpha_3$ were picked by trial and error so that the observed fast-time dynamics of the simulated $\theta$ was qualitatively similar to the observed experimental strain-gage time series.

The following parameters were chosen for simulation:

$\mu = 0.088$, $\alpha_1 = 2.6558$, $\alpha_3 = 0.8805$, $\Omega = 0.85$, $f = 1.8$, $\eta = 6$, $L_0 = 10$, $\kappa = 0.01$, $r = 8$, $\gamma = 1$, and $\epsilon = 2 \times 10^{-5}$.

The forcing amplitude was selected to ensure that the system was chaotic throughout the parameter range (see power spectra in Fig. 8). We first simulated the equations for a period of time with $\phi = 8$ and $\epsilon = 0$ to eliminate transients. Then, the simulation was continued with $\epsilon = 2 \times 10^{-5}$ for $2.5 \times 10^5$ time units ($2.5 \times 10^6$ data points were recorded). At the end of the simulation $\phi$ reached a near-zero value.

Average mutual information and false nearest neighbors (Abarbanel, 1995) were used to select a delay $\tau$ of 7 sample steps, and an embedding dimension $d = 5$. The first $2^{14}$ data points of the scalar $\theta$ data set were used for the reference model. Fig. 9 shows two-dimensional projections of the original ($\theta$ and $\theta$) and the delay-reconstructed ($\theta_i$ and $\theta_i+\tau$) phase spaces. Observe the qualitative similarity between the two graphs.

For the tracking metric $e_7$ (see Eq. (26)) the whole reconstructed data was divided into consecutive data sets of size $2^{12}$ points each (i.e., $M = 2^{12}$). A plot of raw (unscaled) results showing the model RMS error $e_7$ vs. the square of the averaged battery voltage (over the corresponding data set), i.e., $(\tilde{\phi})^2$, is shown in Fig. 10. The figure shows that the tracking metric computed by the algorithm is well described by an affine function of $(\tilde{\phi})^2$. We will discuss this quadratic dependence later, when we analyze the system equations.

For tracking purposes, we calculated the moving average of 10 consecutive points of the tracking metric, along with the standard deviations of these 10 point samples to represent the expected statistical fluctuations. We also used a suitable affine transformation of tracking metric to scale it so that its range corresponded to the actual range spanned by $\phi^2$. Then, the tracking metric and the correct value of $\phi$
were plotted on the same graph (Fig. 11). In the figure, the black dots and outline represent the moving average of the tracking metric along with its local variability, respectively. The thick grey line depicts the corresponding local averages of the battery voltage variable $\phi$.

Fig. 10 (bottom, right corner) and Fig. 11 (top, left corner) show that the tracking metric has some initial insensitivity. This is due to the fact that the local linear models are imperfect, and there is some initial error in prediction even for the reference data set. We cannot detect changes in the damage variable that cause true prediction errors (Eq. 25a) smaller than the inherent modeling error (Eq. 25c).

The preceding results show empirically that changes in the slowly evolving parameter can be tracked with reasonable accuracy by using even a simple tracking metric, namely the RMS $k$-step prediction error (here, $k = 7$).

We now seek some more rigorous and analytical understanding of how and why the tracking method worked for this particular system. The analysis presented below can be carried out because in this case the exact equations governing the system are available. The match between analytical predictions and the results obtained from the tracking algorithm then allow us to use the algorithm with greater confidence even for systems where the governing equations are not, in fact, available.

**AVERAGING OF THE MODEL EQUATIONS AND COMPARISON WITH TRACKING**

Cusumano and Chatterjee (1999) developed basic elements of a qualitative dynamics of systems with slowly evolving damage. They discussed the idea of obtaining *slow flow* equations describing a slowly evolving damage process through averaging of the fast-time equations. The experimental tracking procedure discussed in this paper is, in some sense, a physical realization of the same averaging idea. To show this connection, we now derive the slow flow equations describing the evolution of $\phi$. The discussion of averaging given below follows closely the procedure described in Cusumano and Chatterjee (1999).

We consider again a system of the form of Eqs. (1), but with scalar quantities $g \equiv g$, $\mu \equiv \mu$, and $\phi \equiv \phi$, as in the case studied here. Let $X_0(x_0, \mu(\phi_0), t)$ be the solution to the unperturbed Eq. (1a) with $\epsilon = 0$, $\phi = \phi_0$, and initial conditions $x(t_0) = x_0$. We insert $X_0$ into the right hand
Fig. 11: Tracking results. The solid thick grey line is the local (moving) average of $\Phi$. Black dots show the overlay of scaled tracking metric over the battery voltage curve and light gray outline is drawn at $\pm$ one standard deviation to represent statistical fluctuation in local mean.

side of Eq. (1b), and define
\[
\bar{g} := \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0 + T} g(\mathbf{X}_0(\mathbf{x}_0, \phi, t), \mu(\phi), t) \, dt ,
\]
(28)
where $\phi_0$ in $\mathbf{X}_0$ has been replaced by the slowly drifting quantity $\phi$. We assume that $\mathbf{X}_0$, the solution to the unperturbed or non-drifting fast-time equation, has transients which decay rapidly, and converges to some well-defined steady-state behavior. Under this assumption, the integral above is independent of $t_0$ and $\mathbf{x}_0$, and $\bar{g}$ depends only on $\phi$. We write,
\[
\dot{\phi} = \epsilon \bar{g}(\phi).
\]
(29)
The averaged Eq. (29) approximates the evolution of $\phi$ wherever the assumptions about $\mathbf{X}_0$ are correct.

We now compare the predictions of this slow flow equation obtained from the model of Eqs. (14) with the true evolution of $\phi$ (as obtained from numerical simulations).

The complete equations for the system used in the simulations, Eqs. (10) and (11), are nonlinear, and the response is in general chaotic, so we do not have a closed form expression for the function $\mathbf{X}_0$ to use in the averaging procedure. However, the fact that $\kappa = 0.01$ is small allows us to use the reduced system of Eqs. (14). Under the quasi-steady-state assumption on $\psi$ used to derive Eqs. (14), we find that explicit calculation of the averaged vector field is not needed since Eq. (14b) is already in the form of the slow flow equation:
\[
\bar{g} \equiv -\frac{\phi}{r} \left(1 + \gamma(\phi - \eta)^2\right) ,
\]
(30)
which, from Eq. (29), gives:
\[
\ln \left[ \frac{\phi}{\sqrt{1 + \gamma(\phi - \eta)^2}} \right] + \eta \sqrt{\gamma} \arctan(\sqrt{\gamma}(\phi - \eta)) = (1 + \gamma\eta^2)(-\epsilon t/r + c).
\]
(31)
In the above solution, $c$ is determined by the initial condition $\phi(0) = \phi_0$. For our set of parameters $c = 0.214$. Fig. 12 compares the solution of the slow flow equation with the $\phi$ obtained from numerical simulations on the exact system. The match is good to $\mathcal{O}(\kappa)$, as expected.

We finally connect the analytical equations governing the system evolution to the results from the tracking procedure. Since $\mu(\phi) = \kappa \phi^2/r$ is a small quantity, it is clear that the tracking procedure should recapture the drift in $\phi^2$: this was precisely found be the case (see Fig. 10).

CONCLUSIONS

In this paper we have derived a mathematical model of an experimental system originally used to develop a state-space-based hidden variable tracking method. In the previous experimental work we have shown how the drift observer obtained with the method appeared to track the local mean of the independently measured “hidden” battery voltage. Numerical simulations and mathematical analysis of the governing differential equations were used
in this paper to show exactly how the tracking metric is related to the underlying drifting variable. We have describe how, in general, the analytical treatment of drift in systems such as those studied here will involve averaging of the underlying system equations: though the system studied here permits a simpler analytical treatment, it still demonstrates the main idea that the drift observer generated by the algorithm tracks the solutions to the hidden variable’s slow flow equation, as was demonstrated by comparing the tracking metric (drift observable) to the solutions to the slow flow equation and simulations results for the exact system. The explicit connection made in this paper between hidden variable drift, parameter drift, and drift observable gives us confidence that in other situations where an explicit model is not available the observable generated by the tracking algorithm can still be used with confidence.

REFERENCES


